A comparative study of multigranulation rough sets and concept lattices via rule acquisition

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Abstract

Recently, by combining rough set theory with granular computing, pessimistic and optimistic multigranulation rough sets have been proposed to derive "AND" and "OR" decision rules from decision systems. At the same time, by integrating granular computing and formal concept analysis, Wille's concept lattice and object-oriented concept lattice were used to obtain granular rules and disjunctive rules from formal decision contexts. So, the problem of rule acquisition can bring rough set theory, granular computing and formal concept analysis together. In this study, to shed some light on the comparison and combination of rough set theory, granular computing and formal concept analysis, we investigate the relationship between multigranulation rough sets and concept lattices via rule acquisition. Some interesting results are obtained in this paper: 1) "AND" decision rules in pessimistic multigranulation rough sets are proved to be granular rules in concept lattices, but the inverse may not be true; 2) the combination of the truth parts of an "OR" decision rule in optimistic multigranulation rough sets is an item of the decomposition of a disjunctive rule in concept lattices; 3) a non-redundant disjunctive rule in concept lattices is shown to be the multi-combination of the truth parts of "OR" decision rules in optimistic multigranulation rough sets; 4) the same rule is defined with a same certainty factor but a different support factor in multigranulation rough sets and concept lattices. Moreover, algorithm complexity analysis is made for the acquisition of "AND" decision rules, "OR" decision rules, granular rules and disjunctive rules.

Keywords: Rough set theory; Granular computing; Multigranulation rough set; Concept lattice; Rule acquisition

1. Introduction

Rough set theory, presented by Pawlak [42], has drawn many attentions from researchers over the past thirty-three years [20, 26, 43, 78, 85, 86]. As is well known, its original idea is to partition the universe of discourse into disjoint subsets by a given equivalence or indiscernibility relation. Furthermore, these obtained disjoint subsets are viewed as the basic knowledge which is used to characterize any target set by means of the so-called lower and upper approximations.

Since the equivalence or indiscernibility relation has its limitations in dealing with *information systems* with fuzzy, continuous-valued or interval-valued attributes, the classical rough sets have been generalized and developed by some scholars [14, 16, 18, 57, 59, 73, 77]. Note that the generalized and developed rough sets are beneficial to the implementation of rule acquisition in different kinds of *decision systems* [6, 8, 11, 24, 27, 88, 89].

From the aspect of *granular computing* presented by Zadeh [83] and further elaborated by other researchers [2, 44, 45, 52, 74], the aforementioned generalized and developed rough sets describe a target set by the lower and upper

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approximations under one granulation. However, in the real world, multiple granulations are sometimes required to approximate a target set as well. For example, multi-scale data sets need multiple granulations for set approximations [66], and multi-source data sets inspire Qian et al. [48, 49] to put forward pessimistic multigranulation rough sets and optimistic multigranulation rough sets for applying multi-source information fusion. These information fusion strategies were soon extended to the cases of incomplete, neighborhood, covering and fuzzy environments [17, 36, 37, 55, 68, 71]. Moreover, it deserves to be mentioned that the pessimistic and optimistic multigranulation rough sets were used in [48, 49] to derive "AND" and "OR" decision rules from decision systems, which was further discussed by Yang et al. [70] in terms of local and global measurements of the "AND" and "OR" decision rules.

Formal concept analysis, presented by Wille [64] in the same year as rough set theory, has attracted many researchers [4, 56, 60, 84, 87] to this promising field. Up to now, its applications cover data mining [1, 13], knowledge discovery [10, 46, 69], machine learning [23], software engineering [53], etc. Within this theory, *formal contexts*, *formal concepts* and *concept lattices* are three basic notions for data analysis.

Also, it deserves to be mentioned that in recent years both multigranulation rough sets and concept lattices have been connected with *three-way decisions* whose unified framework description and superiority were given by Yao [78, 79, 80] and whose further investigations and applications were studied by many scholars [9, 15, 19, 28, 35, 39, 72, 82]. For example, Qian et al. [50] established a new decision-theoretic rough set [81] from the perspective of multigranulation rough sets. By taking the intension part of a formal concept as an orthopair [7], Qi et al. [47] put forward *three-way concept lattice* and discussed some useful properties. In addition, Li et al. [30] proposed another three-way concept lattice with the name of *approximate concept lattice* under the environment of incomplete data, where the intension part of a formal concept was expressed as a nested pair which is in fact equivalent to an orthopair.

Recently, more and more attention [12, 21, 22, 25, 54, 62, 75] has been paid to comparing and combining rough set theory and formal concept analysis. Under such a circumstance, object-oriented concept lattice was introduced in [76] by incorporating lower and upper approximation ideas into concept-forming operators and it was further elaborated in [41, 61]. Ren et al. [51] presented the notion of a disjunctive rule in *formal decision contexts* [86] by the help of Wille's and object-oriented concept lattices. Moreover, covering-based rough sets and concept lattices were related to each other in [5, 58] from the viewpoints of approximation operators and reduction. In the meanwhile, integrating formal concept analysis with granular computing has also attracted many researchers [40, 63]. For instance, Wu et al. [65] put forward the notion of a granular rule in formal decision contexts. Li et al. [29] discussed the relation between granular rules and *decision rules* [31]. In addition, rough set theory has been related to granular computing [48, 49, 66, 74], and vice versa.

What is more, the comparison and combination of rough set theory, granular computing and formal concept analysis has received much attention in knowledge representation and discovery [32, 65, 67, 75]. The main contributions of the existing literature in this aspect can be summarized as follows: (1) rough set theory, granular computing and formal concept analysis were combined to form composite concept-forming operators [75] and induce granular concepts [65]; (2) they were jointly used to recognize cognitive concepts by two-step learning approaches [32, 67]. However, little attention has been paid to comparing and combining these three theories from the perspective of rule acquisition. This problem deserves to be investigated since it can not only shed some light on the comparison and combination of them, but also help us to make better decision analysis of the data.

Motivated by the above problem, this study mainly focuses on the comparison and combination of rough set theory, granular computing and formal concept analysis from the viewpoint of rule acquisition. More specifically, we put forward an effective way of transforming decision systems into formal decision contexts and discuss some useful properties. And then the relationship between multigranulation rough sets and concept lattices is analyzed from the perspectives of differences and relations between rules, support and certainty factors for rules, and algorithm complexity analysis of rule acquisition.

The rest of this paper is organized as follows. In Section 2, we recall the notions of Pawlak's rough set, pessimistic multigranulation rough set and optimistic multigranulation rough set as well as their induced "AND" and "OR" decision rules. Moreover, Wille's concept lattice, object-oriented concept lattice, granular rules and disjunctive rules are introduced. In Section 3, we transform decision systems into formal decision contexts and discuss some useful properties. In Section 4, the relationship between multigranulation rough sets and concept lattices is analyzed from the viewpoints of differences and relations between rules, support and certainty factors for rules, and algorithm complexity analysis of the acquisition of "AND" decision rules, "OR" decision rules, granular rules and disjunctive rules. Section 5 concludes this paper with a brief summary and an outlook for further research.

2. Preliminaries

In this section, we review some basic notions such as information system, Pawlak's rough set, pessimistic multigranulation rough sets, "AND" decision rules, optimistic multigranulation rough sets, "OR" decision rules, formal context, Wille's concept lattice, object-oriented concept lattice, granular rules and disjunctive rules.

2.1. Pawlak's rough set

Let U be a non-empty finite set of objects and AT be a non-empty finite set of attributes. Then an information system is considered as a pair $S = (U, AT)$ [42], where the value of $x \in U$ under attribute $a \in AT$ is denoted by $a(x)$.

Given $A \subseteq AT$, an equivalence relation $IND(A)$ is defined as

$$IND(A) = \{(x, y) \in U \times U : \forall a \in A, a(x) = a(y)\}, \tag{1}$$

which partitions U into equivalence classes $[x]_A = \{y \in U : (x, y) \in IND(A)\}$. This partition $\{[x]_A : x \in U\}$ is often denoted by $U/IND(A)$. For a subset X of U,

$$\underline{A}(X) = \{x \in U : [x]_A \subseteq X\} \text{ and } \overline{A}(X) = \{x \in U : [x]_A \cap X \neq \emptyset\} \tag{2}$$

are called the lower and upper approximations [42], respectively. The ordered pair $[\underline{A}(X), \overline{A}(X)]$ is said to be Pawlak's rough set of X with respect to A.

2.2. Multigranulation rough sets and their induced rules

Different from Pawlak's rough set model, multigranulation rough sets were established on the basis of a family of equivalence relations rather than a single one only.

Definition 1 ***[48].*** Let S be an information system and $A_1, A_2, \cdots, A_s \subseteq AT$. Then the pessimistic multigranulation lower and upper approximations of a subset X of U are respectively defined as

$$\underline{\sum_{j=1}^{s} A_j^P}(X) = \{x \in U : [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \cdots \wedge [x]_{A_s} \subseteq X\} \text{ and } \overline{\sum_{j=1}^{s} A_j^P}(X) = \sim \underline{\sum_{j=1}^{s} A_j^P}(\sim X), \tag{3}$$

where $[x]_{A_j}$ $(1 \leq j \leq s)$ is the equivalence class of x in terms of A_j, $\wedge$ is the logical conjunction operator, and $\sim X$ is the complement of X with respect to U.

The pair $\left[\underline{\sum_{j=1}^{s} A_j^P}(X), \overline{\sum_{j=1}^{s} A_j^P}(X)\right]$ is referred to as the pessimistic multigranulation rough set of X with respect to the attribute sets $A_1, A_2, \cdots, A_s$.

In what follows, we discuss rules induced by the pessimistic multigranulation rough sets. Before embarking on this issue, we introduce the notion of a decision system.

A decision system is an information system such that $S = (U, AT \cup D)$ in which AT and D are called the conditional and decision attribute sets, respectively. In this paper, to simplify the subsequent discussion, we only consider D as a singleton set $\{d\}$. As a result, the decision system to be discussed can be represented by $S = (U, AT \cup \{d\})$. For convenience, the partition of U induced by the decision attribute d is denoted by $U/IND(\{d\}) = \{[x]_{\{d\}} : x \in U\}$.

Furthermore, we review the description or semantic explanation [24, 43] of an equivalence class $[x]_{A_j}$ which is denoted by $\mathrm{des}([x]_{A_j})$ and described as $\bigwedge_{a \in A_j} (a, a(x))$, where $a(x)$ is the value of x under attribute a. That is,

$\mathrm{des}([x]_{A_j}) = \bigwedge_{a \in A_j} (a, a(x))$. Similarly, the description or semantic explanation of the equivalence class $[x]_{\{d\}}$ is described as $\mathrm{des}([x]_{\{d\}}) = (d, d(x))$.

According to Definition 1, for any $x \in \underline{\sum_{j=1}^{s} A_j^P}([x]_{\{d\}})$, it generates the following so-called "AND" decision rule from a decision system:

$$r_x^{\wedge} : \bigwedge_{j=1}^{s} (\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})),$$

where every decision rule $\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})$ is true based on Eq. (3). Therefore, $r_x^{\wedge}$ can be represented by

$$r_x^{\wedge} : \left(\bigwedge_{j=1}^{s} \mathrm{des}([x]_{A_j}) \right) \to \mathrm{des}([x]_{\{d\}}). \tag{4}$$

Definition 2 ***[49].*** Let S be an information system and $A_1, A_2, \cdots, A_s \subseteq AT$. Then the optimistic multigranulation lower and upper approximations of a subset X of U are respectively defined as

$$\underline{\sum_{j=1}^{s} A_j^O}(X) = \{x \in U : [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \cdots \vee [x]_{A_s} \subseteq X\} \text{ and } \overline{\sum_{j=1}^{s} A_j^O}(X) = \sim \underline{\sum_{j=1}^{s} A_j^O}(\sim X), \tag{5}$$

where $\vee$ is the logical disjunction operator.

The pair $\left[\underline{\sum_{j=1}^{s} A_j^O}(X), \overline{\sum_{j=1}^{s} A_j^O}(X) \right]$ is referred to as the optimistic multigranulation rough set of X with respect to the attribute sets $A_1, A_2, \cdots, A_s$.

According to Definition 2, for any $x \in \underline{\sum_{j=1}^{s} A_j^O}([x]_{\{d\}})$, it induces the following so-called "OR" decision rule from a decision system:

$$r_x^{\vee} : \bigvee_{j=1}^{s} (\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})), \tag{6}$$

where at least one decision rule $\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})$ is true based on Eq. (5).

2.3. Wille's concept lattice and object-oriented concept lattice

In accordance with the notations in the previous subsections, we denote the object set by U and the attribute set by A. Then, a formal context can be considered as a triple (U, A, I) [64], where $(x, a) \in I$ represents that $x \in U$ possesses $a \in A$ while $(x, a) \notin I$ means the opposite.

In fact, a formal context is a special information system with two-valued input data [34, 38]. To derive Wille's and object-oriented concept lattices, the following four operators are needed: for any $X \subseteq U$ and $B \subseteq A$,

$$\begin{aligned} X^{\uparrow} &= \{a \in A : \forall x \in X, (x, a) \in I\}, \\ B^{\downarrow} &= \{x \in U : \forall a \in B, (x, a) \in I\}, \\ X^{\Box} &= \{a \in A : Ia \subseteq X\}, \\ B^{\Diamond} &= \{x \in U : xI \cap B \neq \emptyset\}, \end{aligned} \tag{7}$$

where $Ia = \{x \in U : (x, a) \in I\}$ and $xI = \{a \in A : (x, a) \in I\}$. Moreover, it is interesting to clarify $Ia = \{a\}^{\downarrow} = \{a\}^{\Diamond}$.

Definition 3 ***[64, 76].*** Let (U, A, I) be a formal context, $X \subseteq U$ and $B \subseteq A$. If $X^{\uparrow} = B$ and $B^{\downarrow} = X$, then (X, B) is called a Wille's concept; if $X^{\Box} = B$ and $B^{\Diamond} = X$, then (X, B) is called an object-oriented concept. For each of the cases, X and B are referred to as the extent and intent of (X, B), respectively.

Proposition 1 *[64, 76].* Let (U, A, I) be a formal context. For $X, X_1, X_2 \subseteq U$ and $B, B_1, B_2 \subseteq A$, the following properties hold:

(i) $X_1 \subseteq X_2 \Rightarrow X_2^{\uparrow} \subseteq X_1^{\uparrow}, X_1^{\Box} \subseteq X_2^{\Box}$;
(ii) $B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow}, B_1^{\Diamond} \subseteq B_2^{\Diamond}$;
(iii) $X \subseteq X^{\uparrow\downarrow}, B \subseteq B^{\downarrow\uparrow}, X^{\Box\Diamond} \subseteq X, B \subseteq B^{\Diamond\Box}$;
(iv) $(X_1 \cup X_2)^{\uparrow} = X_1^{\uparrow} \cap X_2^{\uparrow}, (X_1 \cap X_2)^{\Box} = X_1^{\Box} \cap X_2^{\Box}$;
(v) $(B_1 \cup B_2)^{\downarrow} = B_1^{\downarrow} \cap B_2^{\downarrow}, (B_1 \cup B_2)^{\Diamond} = B_1^{\Diamond} \cup B_2^{\Diamond}$;
(vi) $(X^{\uparrow\downarrow}, X^{\uparrow}), (B^{\downarrow}, B^{\downarrow\uparrow})$ are Wille's concepts and $(X^{\Box\Diamond}, X^{\Box}), (B^{\Diamond}, B^{\Diamond\Box})$ are object-oriented concepts.

If Wille's and object-oriented concepts of (U, A, I) are respectively ordered by

$$\begin{aligned}(X_1, B_1) \leq_W (X_2, B_2) &\Longleftrightarrow X_1 \subseteq X_2\ (\Longleftrightarrow B_2 \subseteq B_1),\\ (X_1, B_1) \leq_O (X_2, B_2) &\Longleftrightarrow X_1 \subseteq X_2\ (\Longleftrightarrow B_1 \subseteq B_2),\end{aligned} \tag{8}$$

both of them form complete lattices which are called Wille's concept lattice [64] and object-oriented concept lattice [76], respectively. Hereinafter, the former is denoted by $\underline{\mathfrak{B}}_W(U, A, I)$ and the latter is denoted by $\underline{\mathfrak{B}}_O(U, A, I)$.

2.4. *Granular rules and disjunctive rules*

A formal decision context (also called decision formal context) [86] is a quintuple $\Pi = (U, A, I, D, J)$ with (U, A, I) and (U, D, J) being formal contexts, where A and D with $A \cap D = \emptyset$ are called the conditional and decision attribute sets, respectively.

To avoid confusion, the operators $\uparrow$ and $\downarrow$ defined in Eq. (7) are expressed by different forms when they appear in different contexts of Π. Concretely, in the context (U, A, I), the notations $\uparrow$ and $\downarrow$ are reserved in their current forms, while in the context (U, D, J), they are changed as $\Uparrow$ and $\Downarrow$, respectively. For brevity, sometimes we write $\{x\}^{\uparrow}$ as $x^{\uparrow}$, $\{a\}^{\downarrow}$ as $a^{\downarrow}$, $\{x\}^{\Uparrow}$ as $x^{\Uparrow}$, and $\{a\}^{\Downarrow}$ as $a^{\Downarrow}$.

Definition 4 *[65].* Let Π be a formal decision context and $x \in U$. If $x^{\uparrow\downarrow} \subseteq x^{\Uparrow\Downarrow}$, then $x^{\uparrow} \rightarrow x^{\Uparrow}$ is called a granular rule.

A granular rule $x^{\uparrow} \rightarrow x^{\Uparrow}$ says that each object having all the conditional attributes in $x^{\uparrow}$ also has all the decision attributes in $x^{\Uparrow}$. That is, if $\wedge x^{\uparrow}$, then $\wedge x^{\Uparrow}$, where $\wedge$ is the logical conjunction operator.

Definition 5 *[51].* Let Π be a formal decision context, $\underline{\mathfrak{B}}_O(U, A, I)$ be the object-oriented concept lattice of (U, A, I) and $\underline{\mathfrak{B}}_W(U, D, J)$ be Wille's concept lattice of (U, D, J). For $(X, B) \in \underline{\mathfrak{B}}_O(U, A, I)$ and $(Y, C) \in \underline{\mathfrak{B}}_W(U, D, J)$, if X, B, Y, C are all non-empty and $X \subseteq Y$, then the expression $B \rightarrow_{disj} C$ is called a disjunctive rule.

Hereinafter, a disjunctive rule $B \rightarrow_{disj} C$ is rewritten as $B \rightarrow C$ when there is no confusion. A disjunctive rule says that each object having at least one conditional attribute in B has all the decision attributes in C. That is, if $\vee B$, then $\wedge C$, where $\vee$ is the logical disjunction operator.

Definition 6 *[51].* Let Π be a formal decision context. For two disjunctive rules $B_1 \rightarrow C_1$ and $B_2 \rightarrow C_2$, if $B_2 \subseteq B_1$ and $C_2 \subseteq C_1$, we say that $B_2 \rightarrow C_2$ can be implied by $B_1 \rightarrow C_1$. Furthermore, let Ω be a set of disjunctive rules. For $B \rightarrow C \in \Omega$, if there exists another disjunctive rule $B_0 \rightarrow C_0 \in \Omega$ such that $B_0 \rightarrow C_0$ implies $B \rightarrow C$, we say that $B \rightarrow C$ is redundant in Ω; otherwise, it is said to be non-redundant in Ω.

Generally speaking, it is more appealing to derive non-redundant disjunctive rules from a formal decision context.

3. Transforming decision systems into formal decision contexts

In this section, we focus on transforming decision systems into formal decision contexts and discuss some useful properties to facilitate our subsequent discussion.

Let $S = (U, AT \cup \{d\})$ be a decision system with $U = \{x_1, x_2, \cdots, x_m\}$ and $AT = \{a_1, a_2, \cdots, a_n\}$. Note that S can be represented as a two-dimensional table (see Table 1 for details), where $a_j(x_i)$ $(1 \leq i \leq m, 1 \leq j \leq n)$ is the value of x_i under the conditional attribute a_j and $d(x_i)$ is the value of x_i under the decision attribute d.

Table 1: A decision system $S = (U, AT \cup \{d\})$

	a_1	a_2	$\cdots$	a_n	d
x_1	$a_1(x_1)$	$a_2(x_1)$	$\cdots$	$a_n(x_1)$	$d(x_1)$
x_2	$a_1(x_2)$	$a_2(x_2)$	$\cdots$	$a_n(x_2)$	$d(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	$a_1(x_m)$	$a_2(x_m)$	$\cdots$	$a_n(x_m)$	$d(x_m)$

Generally speaking, for a conditional attribute a_j, there may exist a repetition of values in the array $a_j(x_1)$, $a_j(x_2)$, $\cdots$, $a_j(x_m)$. For our purpose, we create a new array v_1^j, v_2^j, $\cdots$, $v_{m_j}^j$ without the repetition of values. In other words, v_1^j, v_2^j, $\cdots$, $v_{m_j}^j$ are parts of $a_j(x_1)$, $a_j(x_2)$, $\cdots$, $a_j(x_m)$ by avoiding repetition. Similarly, for the decision attribute d, from the original array $d(x_1)$, $d(x_2)$, $\cdots$, $d(x_m)$, we can also create a new array v_1^d, v_2^d, $\cdots$, $v_{m_d}^d$ without the repetition of values. That is, v_1^d, v_2^d, $\cdots$, $v_{m_d}^d$ are non-repeatedly selected from $d(x_1)$, $d(x_2)$, $\cdots$, $d(x_m)$.

Then, based on the above convention, we can obtain a formal decision context $\Pi = (U, A, I, D, J)$, where $U = \{x_1, x_2, \cdots, x_m\}$, $A = \{a_1(v_1^1), \cdots, a_1(v_{m_1}^1), a_2(v_1^2), \cdots, a_2(v_{m_2}^2), \cdots, a_n(v_1^n), \cdots, a_n(v_{m_n}^n)\}$, $D = \{d(v_1^d), d(v_2^d), \cdots, d(v_{m_d}^d)\}$. Note that Π can be represented as a two-dimensional table. More details can be found in Table 2. In the table, the Boolean function λ_{ijk} is used to describe whether the object x_i has the conditional attribute $a_j(v_k^j)$ and the Boolean function μ_{ik} is used to describe whether the object x_i has the decision attribute $d(v_k^d)$. More specifically,

$$\lambda_{ijk} = \begin{cases} 1, & \text{if } a_j(x_i) = v_k^j, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

$$\mu_{ik} = \begin{cases} 1, & \text{if } d(x_i) = v_k^d, \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

We say that the formal decision context Π shown in Table 2 is induced by the decision system S.

Table 2: A formal decision context $\Pi = (U, A, I, D, J)$ induced by S

	$a_1(v_1^1)$	$\cdots$	$a_1(v_{m_1}^1)$	$a_2(v_1^2)$	$\cdots$	$a_2(v_{m_2}^2)$	$\cdots$	$a_n(v_1^n)$	$\cdots$	$a_n(v_{m_n}^n)$	$d(v_1^d)$	$\cdots$	$d(v_{m_d}^d)$
x_1	λ_{111}	$\cdots$	λ_{11m_1}	λ_{121}	$\cdots$	λ_{12m_2}	$\cdots$	λ_{1n1}	$\cdots$	λ_{1nm_n}	μ_{11}	$\cdots$	μ_{1m_d}
x_2	λ_{211}	$\cdots$	λ_{21m_1}	λ_{221}	$\cdots$	λ_{22m_2}	$\cdots$	λ_{2n1}	$\cdots$	λ_{2nm_n}	μ_{21}	$\cdots$	μ_{2m_d}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	λ_{m11}	$\cdots$	λ_{m1m_1}	λ_{m21}	$\cdots$	λ_{m2m_2}	$\cdots$	λ_{mn1}	$\cdots$	λ_{mnm_n}	μ_{m1}	$\cdots$	μ_{mm_d}

Proposition 2. Let $\Pi = (U, A, I, D, J)$ be the formal decision context induced by S, where $U = \{x_1, x_2, \cdots, x_m\}$, $A = \{a_1(v_1^1), \cdots, a_1(v_{m_1}^1), a_2(v_1^2), \cdots, a_2(v_{m_2}^2), \cdots, a_n(v_1^n), \cdots, a_n(v_{m_n}^n)\}$ and $D = \{d(v_1^d), d(v_2^d), \cdots, d(v_{m_d}^d)\}$. Then, for any $x_i \in U$,

(i) there exists $a_j(v_k^j)$ $(\forall j \in \{1, 2, \cdots, n\})$ such that $(x_i, a_j(v_k^j)) \in I$ and $(x_i, a_j(v_t^j)) \notin I$ for all $t \in \{1, 2, \cdots, m_j\} - \{k\}$;

(ii) there exists $d(v_k^d)$ such that $(x_i, d(v_k^d)) \in J$ and $(x_i, d(v_t^d)) \notin J$ for all $t \in \{1, 2, \cdots, m_d\} - \{k\}$.

Proof. (i) Suppose $a_j(x_i)$ is the value of x_i under the conditional attribute a_j in the decision system S. Then based on the above discussion of transforming S into Π, there exists v_k^j such that $a_j(x_i) = v_k^j$ and $a_j(x_i) \neq v_t^j$ for all $t \in \{1, 2, \cdots, m_j\} - \{k\}$. By Eq. (9), it follows that $\lambda_{ijk} = 1$ and $\lambda_{ijt} = 0$ for all $t \in \{1, 2, \cdots, m_j\} - \{k\}$, which leads to $(x_i, a_j(v_k^j)) \in I$ and $(x_i, a_j(v_t^j)) \notin I$ for all $t \in \{1, 2, \cdots, m_j\} - \{k\}$.

(ii) It can be proved in a manner similar to (i). □

Proposition 3. Let $\Pi = (U, A, I, D, J)$ be the formal decision context induced by S, where $U = \{x_1, x_2, \cdots, x_m\}$, $A = \{a_1(v_1^1), \cdots, a_1(v_{m_1}^1), a_2(v_1^2), \cdots, a_2(v_{m_2}^2), \cdots, a_n(v_1^n), \cdots, a_n(v_{m_n}^n)\}$ and $D = \{d(v_1^d), d(v_2^d), \cdots, d(v_{m_d}^d)\}$. Then, for any $x_i \in U$,

(i) there exists $a_j(v_k^j)$ ($\forall j \in \{1, 2, \cdots, n\}$) such that $\{a_j(v_k^j)\}^{\downarrow} = [x_i]_{\{a_j\}}$, where $[x_i]_{\{a_j\}}$ is the equivalence class of x_i in terms of $\{a_j\}$, and $^{\downarrow}$ is the operator defined in Eq. (7);

(ii) there exists $d(v_k^d)$ such that $\{d(v_k^d)\}^{\Downarrow} = [x_i]_{\{d\}}$, where $[x_i]_{\{d\}}$ is the equivalence class of x_i in terms of $\{d\}$, and $^{\Downarrow}$ is the operator specified in Section 2.4.

Proof. (i) Suppose $a_j(x_i)$ is the value of x_i under the conditional attribute a_j in the decision system S. Then based on the discussion of transforming S into Π, there exists v_k^j such that $a_j(x_i) = v_k^j$. By Eqs. (1), (7) and (9), we conclude $\{a_j(v_k^j)\}^{\downarrow} = \{x \in U : (x, a_j(v_k^j)) \in I\} = \{x \in U : a_j(x) = v_k^j\} = \{x \in U : a_j(x) = a_j(x_i)\} = [x_i]_{\{a_j\}}$.

(ii) It can be proved in a manner similar to (i). □

Finally, we use a real example to show the process of transforming a decision system into a formal decision context.

Example 1. Table 3 depicts a dataset of five patients who suffer (or do not suffer) from flu.

Table 3: A flu dataset

Patient	Temperature	Headache	Flu
x_1	Normal	No	No
x_2	Slightly high	A little	No
x_3	Slightly high	No	No
x_4	High	A little	Yes
x_5	Normal	Serious	Yes

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $AT = \{a_1, a_2\}$ (a_1: Temperature, a_2: Headache) and d: Flu. Then we can establish a decision system $S = (U, AT \cup \{d\})$. From Table 3, we obtain the following information:

$$[x_1]_{\{a_1\}} = [x_5]_{\{a_1\}} = \{x_1, x_5\}, \quad [x_2]_{\{a_1\}} = [x_3]_{\{a_1\}} = \{x_2, x_3\}, \quad [x_4]_{\{a_1\}} = \{x_4\},$$

$$[x_1]_{\{a_2\}} = [x_3]_{\{a_2\}} = \{x_1, x_3\}, \quad [x_2]_{\{a_2\}} = [x_4]_{\{a_2\}} = \{x_2, x_4\}, \quad [x_5]_{\{a_2\}} = \{x_5\},$$

$$[x_1]_{\{d\}} = [x_2]_{\{d\}} = [x_3]_{\{d\}} = \{x_1, x_2, x_3\}, \quad [x_4]_{\{d\}} = [x_5]_{\{d\}} = \{x_4, x_5\}.$$

By the above transforming approach, we generate a formal decision context $\Pi = (U, A, I, D, J)$ with $A = \{a_1(\text{Normal}), a_1(\text{Slightly high}), a_1(\text{High}), a_2(\text{No}), a_2(\text{A little}), a_2(\text{Serious})\}$ and $D = \{d(\text{No}), d(\text{Yes})\}$. The binary relations I and J are shown in Table 4. In the table, number 1 in the i-th row and j-th column means that the i-th object has the j-th attribute, and number 0 means the opposite.

Table 4: A formal decision context $\Pi = (U, A, I, D, J)$ induced by S

Patient	a_1(Normal)	a_1(Slightly high)	a_1(High)	a_2(No)	a_2(A little)	a_2(Serious)	d(No)	d(Yes)
x_1	1	0	0	1	0	0	1	0
x_2	0	1	0	0	1	0	1	0
x_3	0	1	0	1	0	0	1	0
x_4	0	0	1	0	1	0	0	1
x_5	1	0	0	0	0	1	0	1

From Table 4, we get the following information:

$$\{a_1(\text{Normal})\}^{\downarrow} = \{x_1, x_5\}, \quad \{a_1(\text{Slightly high})\}^{\downarrow} = \{x_2, x_3\}, \quad \{a_1(\text{High})\}^{\downarrow} = \{x_4\},$$

$$\{a_2(\text{No})\}^{\downarrow} = \{x_1, x_3\}, \quad \{a_2(\text{A little})\}^{\downarrow} = \{x_2, x_4\}, \quad \{a_2(\text{Serious})\}^{\downarrow} = \{x_5\},$$
$$\{d(\text{No})\}^{\Downarrow} = \{x_1, x_2, x_3\}, \quad \{d(\text{Yes})\}^{\Downarrow} = \{x_4, x_5\}.$$

Then, it is easy to check that Propositions 2 and 3 are true for Example 1.

4. A comparative study of multigranulation rough sets and concept lattices via rule acquisition

In this section, the relationship between multigranulation rough sets and concept lattices is analyzed from the perspectives of differences and relations between rules, support and certainty factors for rules and algorithm complexity analysis of rule acquisition.

4.1. Differences and relations between rules in multigranulation rough sets and concept lattices

In what follows, we mainly discuss the relationship between "AND" decision rules and granular rules, and the relationship between "OR" decision rules and disjunctive rules.

4.1.1. Relationship between "AND" decision rules and granular rules

For a decision system $S = (U, AT \cup \{d\})$, let $\underline{\sum_{j=1}^{s} A_j^P}([x]_{\{d\}})$ be the pessimistic multigranulation lower approximation of $[x]_{\{d\}} \in U/IND(\{d\})$ with respect to the attribute sets $A_1, A_2, \cdots, A_s \subseteq AT$.

Theorem 1. Let S be a decision system, $A_1, A_2, \cdots, A_s$ be pairwise different subsets of AT with $\cup A_j = AT$, $x \in \underline{\sum_{j=1}^{s} A_j^P}([x]_{\{d\}})$, and Π be the formal decision context induced by S. Then the "AND" decision rule $\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})$ is a granular rule extracted from Π.

Proof. Since $x \in \underline{\sum_{j=1}^{s} A_j^P}([x]_{\{d\}})$, by Definition 1, we conclude $[x]_{A_j} \subseteq [x]_{\{d\}}$ for all $j = 1, 2, \cdots, s$. Furthermore, combining $\cup A_j = AT$ with (v) of Proposition 1 and Propositions 2 and 3, we have

$$\begin{aligned}
x^{\uparrow\downarrow} &= \{a_1(a_1(x)), a_2(a_2(x)), \cdots, a_{|AT|}(a_{|AT|}(x))\}^{\downarrow} \\
&= \left(\left(\bigcup_{a\in A_1} \{a(a(x))\}\right) \bigcup \left(\bigcup_{a\in A_2} \{a(a(x))\}\right) \bigcup \cdots \bigcup \left(\bigcup_{a\in A_s} \{a(a(x))\}\right)\right)^{\downarrow} \\
&= \left(\bigcup_{a\in A_1} \{a(a(x))\}\right)^{\downarrow} \bigcap \left(\bigcup_{a\in A_2} \{a(a(x))\}\right)^{\downarrow} \bigcap \cdots \bigcap \left(\bigcup_{a\in A_s} \{a(a(x))\}\right)^{\downarrow} \\
&= \left(\bigcap_{a\in A_1} \{a(a(x))\}^{\downarrow}\right) \bigcap \left(\bigcap_{a\in A_2} \{a(a(x))\}^{\downarrow}\right) \bigcap \cdots \bigcap \left(\bigcap_{a\in A_s} \{a(a(x)\}^{\downarrow}\right) \\
&= \left(\bigcap_{a\in A_1} [x]_{\{a\}}\right) \bigcap \left(\bigcap_{a\in A_2} [x]_{\{a\}}\right) \bigcap \cdots \bigcap \left(\bigcap_{a\in A_s} [x]_{\{a\}}\right) \\
&= [x]_{A_1} \cap [x]_{A_2} \cap \cdots \cap [x]_{A_s} \\
&\subseteq [x]_{\{d\}} \\
&= \{d(d(x))\}^{\Downarrow} \\
&= x^{\Uparrow\Downarrow}.
\end{aligned} \tag{11}$$

Then, by Definition 4, we obtain a granular rule $\{a_1(a_1(x)), a_2(a_2(x)), \cdots, a_{|AT|}(a_{|AT|}(x))\} \to \{d(d(x))\}$ which can be expressed as $\bigwedge_{j=1}^{|AT|} a_j(a_j(x)) \to d(d(x))$ and rewritten as $\bigwedge_{j=1}^{s}\left(\bigwedge_{a\in A_j} a(a(x))\right) \to d(d(x))$. Since from the viewpoint of semantic explanation, $a(a(x))$ is equivalent to $(a, a(x))$ and $d(d(x))$ is equivalent to $(d, d(x))$, it follows $\text{des}([x]_{A_j}) = \bigwedge_{a\in A_j} (a, a(x)) = \bigwedge_{a\in A_j} a(a(x))$ and $\text{des}([x]_{\{d\}}) = (d, d(x)) = d(d(x))$. So, the granular rule $\bigwedge_{j=1}^{s}\left(\bigwedge_{a\in A_j} a(a(x))\right) \to d(d(x))$ is just the "AND" decision rule $\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})$. □

Example 2. Continued with Example 1. Take $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$. We can compute the partitions $U/IND(A_1) = \{\{x_1, x_5\}, \{x_2, x_3\}, \{x_4\}\}$, $U/IND(A_2) = \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_5\}\}$, and $U/IND(\{d\}) = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}\}$. By Eq. (3), $\underline{\{A_1 + A_2\}}^P(\{x_1, x_2, x_3\}) = \{x_3\}$ and $\underline{\{A_1 + A_2\}}^P(\{x_4, x_5\}) = \emptyset$. Then, by Eq. (4), we obtain the following "AND" decision rule from the decision system S in Table 3 using pessimistic multigranulation rough sets:

$$r^{\wedge}_{x_3} : \text{(Temperature, Slightly high)} \wedge \text{(Headache, No)} \to \text{(Flu, No)}.$$

On the other hand, based on Definition 4, we generate the following granular rules from the induced formal decision context Π in Table 4:

$$\begin{aligned}
r^g_{x_1} &: \text{If Temperature(Normal) and Headache(No), then Flu(No);}\\
r^g_{x_2} &: \text{If Temperature(Slightly high) and Headache(A little), then Flu(No);}\\
r^g_{x_3} &: \text{If Temperature(Slightly high) and Headache(No), then Flu(No);}\\
r^g_{x_4} &: \text{If Temperature(High) and Headache(A little), then Flu(Yes);}\\
r^g_{x_5} &: \text{If Temperature(Normal) and Headache(Serious), then Flu(Yes).}
\end{aligned}$$

Since $r^{\wedge}_{x_3}$ is just $r^g_{x_3}$, Example 2 confirms that the "AND" decision rule derived from S is the granular rule derived from the induced formal decision context Π.

Theorem 2. Let S be a decision system, $A_1, A_2, \cdots, A_s$ be pairwise different subsets of AT with $\cup A_j = AT$, and Π be the formal decision context induced by S. Then some granular rules extracted from Π may not be "AND" decision rules extracted from S.

We use the following counter-example to confirm our statement in Theorem 2.

Example 3. Continued with Example 2. Note that $A_1, A_2, \cdots, A_s$ are pairwise different subsets of AT with $\cup A_j = AT$. Under such a circumstance, we conclude that s is less than or equal to 3 regardless of the empty set. On the other hand, since our discussion is under the multigranulation environment, s should be greater than or equal to 2. To sum up, we obtain $2 \le s \le 3$.

When $s = 2$, there are three cases regardless of the order:

$$1)\ A_1 = \{a_1\}, A_2 = \{a_2\}, \quad 2)\ A_1 = \{a_1\}, A_2 = \{a_1, a_2\}, \quad 3)\ A_1 = \{a_1, a_2\}, A_2 = \{a_2\};$$

when $s = 3$, there is only one case regardless of the order:

$$4)\ A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_1, a_2\}.$$

Case 1) $A_1 = \{a_1\}$, $A_2 = \{a_2\}$. Based on the results obtained in Example 2, we know that the granular rules $r^g_{x_1}$, $r^g_{x_2}$, $r^g_{x_4}$ and $r^g_{x_5}$ derived from the induced formal decision context Π are not "AND" decision rules extracted from S.

Case 2) $A_1 = \{a_1\}$, $A_2 = \{a_1, a_2\}$. By Eq. (4), we obtain the following "AND" decision rules from S in Table 3 using pessimistic multigranulation rough sets:

$$\begin{aligned}
r^{\wedge}_{x_2} &: \text{(Temperature, Slightly high)} \wedge \text{(Headache, A little)} \to \text{(Flu, No);}\\
r^{\wedge}_{x_3} &: \text{(Temperature, Slightly high)} \wedge \text{(Headache, No)} \to \text{(Flu, No);}\\
r^{\wedge}_{x_4} &: \text{(Temperature, High)} \wedge \text{(Headache, A little)} \to \text{(Flu, Yes).}
\end{aligned}$$

Combining them with the results in Example 2, we know that the granular rules $r^g_{x_1}$ and $r^g_{x_5}$ derived from Π are not "AND" decision rules extracted from S.

Case 3) $A_1 = \{a_1, a_2\}$, $A_2 = \{a_2\}$. By Eq. (4), we obtain the following "AND" decision rules from S in Table 3 using pessimistic multigranulation rough sets:

$$\begin{aligned}
r^{\wedge}_{x_1} &: \text{(Temperature, Normal)} \wedge \text{(Headache, No)} \to \text{(Flu, No);}\\
r^{\wedge}_{x_3} &: \text{(Temperature, Slightly high)} \wedge \text{(Headache, No)} \to \text{(Flu, No);}\\
r^{\wedge}_{x_5} &: \text{(Temperature, Normal)} \wedge \text{(Headache, Serious)} \to \text{(Flu, Yes).}
\end{aligned}$$

Combining them with the results in Example 2, we know that the granular rules $r^g_{x_2}$ and $r^g_{x_4}$ derived from Π are not "AND" decision rules extracted from S.

Case 4) $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_1, a_2\}$. The conclusion is the same as that of Case 1. That is, the granular rules $r^g_{x_1}$, $r^g_{x_2}$, $r^g_{x_4}$ and $r^g_{x_5}$ derived from Π are not "AND" decision rules extracted from S.

In summary, Example 3 verifies our statement in Theorem 2.

4.1.2. Relationship between "OR" decision rules and disjunctive rules

Before embarking on this issue, we introduce the combination of the truth parts of an "OR" decision rule.

Given a decision system $S = (U, AT \cup \{d\})$, let $\underline{\sum_{j=1}^{s} A_j^O}([x]_{\{d\}})$ be the optimistic multigranulation lower approximation of $[x]_{\{d\}} \in U/IND(\{d\})$ with respect to the attribute sets $A_1, A_2, \cdots, A_s \subseteq AT$.

For $x \in \underline{\sum_{j=1}^{s} A_j^O}([x]_{\{d\}})$, the "OR" decision rule

$$r_x^{\vee} : \bigvee_{j=1}^{s} (\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))$$

says that some decision rules $\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})$ are true while others are not. Without loss of generality, we assume that $\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})$ $(1 \leq j \leq k)$ are true. Then, by combining $\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})$ $(1 \leq j \leq k)$, we obtain

$$r_x^{\vee *} : \left(\bigvee_{j=1}^{k} \mathrm{des}([x]_{A_j}) \right) \to \mathrm{des}([x]_{\{d\}}).$$

Hereinafter, we say that $r_x^{\vee *}$ is the combination of the truth parts of the "OR" decision rule $r_x^{\vee}$.

Moreover, we discuss the decomposition of a disjunctive rule.

Let $\Pi = (U, A, I, D, J)$ be a formal decision context, $\underline{\mathfrak{B}}_O(U, A, I)$ be the object-oriented concept lattice of (U, A, I) and $\underline{\mathfrak{B}}_W(U, D, J)$ be Wille's concept lattice of (U, D, J). For $(X, B) \in \underline{\mathfrak{B}}_O(U, A, I)$ with $B = \{b_1, b_2, \cdots, b_l\}$ and $(Y, C) \in \underline{\mathfrak{B}}_W(U, D, J)$ with $C = \{c_1, c_2, \cdots, c_t\}$, the disjunctive rule $B \to C$, represented as $b_1 \vee b_2 \vee \cdots \vee b_l \to c_1 \wedge c_2 \wedge \cdots \wedge c_t$, can be decomposed into

$$\begin{aligned} &b_1 \to c_1 \wedge c_2 \wedge \cdots \wedge c_t, \\ &b_2 \to c_1 \wedge c_2 \wedge \cdots \wedge c_t, \\ &\quad \vdots \\ &b_l \to c_1 \wedge c_2 \wedge \cdots \wedge c_t. \end{aligned}$$

By combining parts of them, we generate a new rule $\vee B_0 \to \wedge C$, where $B_0 \subseteq B$. Hereinafter, we say that $\vee B_0 \to \wedge C$ is an item of the decomposition of the disjunctive rule $B \to C$.

Theorem 3. Let S be a decision system, $A_1, A_2, \cdots, A_s$ be pairwise different singleton subsets of AT with $\cup A_j = AT$, $x \in \underline{\sum_{j=1}^{s} A_j^O}([x]_{\{d\}})$, and Π be the formal decision context induced by S. Then the combination of the truth parts of the "OR" decision rule $\bigvee_{j=1}^{s} (\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))$ is an item of the decomposition of a disjunctive rule derived from Π.

Proof. Suppose

$$r_x^{\vee *} : \left(\bigvee_{j=1}^{k} \mathrm{des}([x]_{A_j}) \right) \to \mathrm{des}([x]_{\{d\}})$$

is the combination of the truth parts of the "OR" decision rule

$$r_x^{\vee} : \bigvee_{j=1}^{s} (\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})).$$

Moreover, since $A_1, A_2, \cdots, A_s$ are pairwise different singleton subsets of AT with $\cup A_j = AT$, we assume $A_1 = \{a_1\}$, $A_2 = \{a_2\}, \cdots, A_s = \{a_s\}$. So, $r_x^{\vee *}$ can be represented as

$$(a_1, a_1(x)) \vee (a_2, a_2(x)) \vee \cdots \vee (a_k, a_k(x)) \rightarrow (d, d(x)). \tag{12}$$

Also, we have

$$[x]_{\{a_j\}} \subseteq [x]_{\{d\}} \ (1 \leq j \leq k). \tag{13}$$

On the other hand, by (vi) of Proposition 1, $(([x]_{\{d\}})^{\Box\Diamond}, ([x]_{\{d\}})^{\Box})$ is an object-oriented concept of (U, A, I) and $(([x]_{\{d\}})^{\Uparrow\Downarrow}, ([x]_{\{d\}})^{\Uparrow})$ is a Wille's concept of (U, D, J). Considering that $([x]_{\{d\}})^{\Uparrow\Downarrow} = \{d(d(x))\}^{\Downarrow} = [x]_{\{d\}}$, we have $([x]_{\{d\}})^{\Box\Diamond} \subseteq ([x]_{\{d\}})^{\Uparrow\Downarrow}$ according to (iii) of Proposition 1. Furthermore, note that $([x]_{\{d\}})^{\Box\Diamond}$, $([x]_{\{d\}})^{\Box}$, $([x]_{\{d\}})^{\Uparrow\Downarrow}$ and $([x]_{\{d\}})^{\Uparrow}$ are all non-empty. Then by Definition 5, we generate a disjunctive rule $([x]_{\{d\}})^{\Box} \rightarrow ([x]_{\{d\}})^{\Uparrow}$. Besides, based on Eqs. (7) and (13) and Proposition 2, we have $a_j(a_j(x)) \in ([x]_{\{d\}})^{\Box}$ for all $1 \leq j \leq k$ because $Ia_j(a_j(x)) = \{a_j(a_j(x))\}^{\downarrow} = [x]_{\{a_j\}} \subseteq [x]_{\{d\}}$. As a result, it follows $\{a_1(a_1(x)), a_2(a_2(x)), \cdots, a_k(a_k(x))\} \subseteq ([x]_{\{d\}})^{\Box}$. So,

$$a_1(a_1(x)) \vee a_2(a_2(x)) \vee \cdots \vee a_k(a_k(x)) \rightarrow d(d(x)) \tag{14}$$

is an item of the decomposition of the disjunctive rule $([x]_{\{d\}})^{\Box} \rightarrow ([x]_{\{d\}})^{\Uparrow}$ due to $([x]_{\{d\}})^{\Uparrow} = d(d(x))$. Since from the viewpoint of semantic explanation, $a_j(a_j(x))$ is equivalent to $(a_j, a_j(x))$ and $d(d(x))$ is equivalent to $(d, d(x))$, we conclude that Eqs. (12) and (14) are the same. □

Example 4. Continued with Example 1. Take $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$. Then, $U/IND(A_1) = \{\{x_1, x_5\}, \{x_2, x_3\}, \{x_4\}\}$, $U/IND(A_2) = \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_5\}\}$, and $U/IND(\{d\}) = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}\}$. By Eq. (5), $\underline{\{A_1 + A_2\}}^{O}(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}$ and $\underline{\{A_1 + A_2\}}^{O}(\{x_4, x_5\}) = \{x_4, x_5\}$. Furthermore, by Eq. (6), we obtain the following "OR" decision rules from the decision system S in Table 3 using optimistic multigranulation rough sets:

$r_{x_1}^{\vee}$: Temperature(Normal) → Flu(No) ∨ Headache(No) → Flu(No);
$r_{x_2}^{\vee}$: Temperature(Slightly high) → Flu(No) ∨ Headache(A little) → Flu(No);
$r_{x_3}^{\vee}$: Temperature(Slightly high) → Flu(No) ∨ Headache(No) → Flu(No);
$r_{x_4}^{\vee}$: Temperature(High) → Flu(Yes) ∨ Headache(A little) → Flu(Yes);
$r_{x_5}^{\vee}$: Temperature(Normal) → Flu(Yes) ∨ Headache(Serious) → Flu(Yes).

By the combination of the truth parts of every "OR" decision rule $r_{x_i}^{\vee}$ $(1 \leq i \leq 5)$, we get

$r_{x_1}^{\vee *}$: Headache(No) → Flu(No);
$r_{x_2}^{\vee *}$: Temperature(Slightly high) → Flu(No);
$r_{x_3}^{\vee *}$: Temperature(Slightly high) ∨ Headache(No) → Flu(No);
$r_{x_4}^{\vee *}$: Temperature(High) → Flu(Yes);
$r_{x_5}^{\vee *}$: Headache(Serious) → Flu(Yes).

According to the formal decision context Π induced by S in Table 4, we can compute

$$\underline{\mathfrak{B}}_O(U,A,I) = \left\{ \begin{array}{l} (\emptyset,\emptyset), \\ (U,A), \\ (\{x_4\},\{a_1(\text{High})\}), \\ (\{x_5\},\{a_2(\text{Serious})\}), \\ (\{x_1,x_3\},\{a_2(\text{No})\}), \\ (\{x_2,x_3\},\{a_1(\text{Slightly high})\}), \\ (\{x_2,x_4\},\{a_1(\text{High}),a_2(\text{A little})\}), \\ (\{x_1,x_5\},\{a_1(\text{Normal}),a_2(\text{Serious})\}), \\ (\{x_4,x_5\},\{a_1(\text{High}),a_2(\text{Serious})\}), \\ (\{x_1,x_2,x_3\},\{a_1(\text{Slightly high}),a_2(\text{No})\}), \\ (\{x_1,x_3,x_4\},\{a_1(\text{High}),a_2(\text{No})\}), \\ (\{x_1,x_3,x_5\},\{a_1(\text{Normal}),a_2(\text{No}),a_2(\text{Serious})\}), \\ (\{x_1,x_4,x_5\},\{a_1(\text{Normal}),a_1(\text{High}),a_2(\text{Serious})\}), \\ (\{x_2,x_3,x_4\},\{a_1(\text{Slightly high}),a_1(\text{High}),a_2(\text{A little})\}), \\ (\{x_2,x_3,x_5\},\{a_1(\text{Slightly high}),a_2(\text{Serious})\}), \\ (\{x_2,x_4,x_5\},\{a_1(\text{High}),a_2(\text{A little}),a_2(\text{Serious})\}), \\ (\{x_1,x_2,x_3,x_4\},\{a_1(\text{Slightly high}),a_1(\text{High}),a_2(\text{No}),a_2(\text{A little})\}), \\ (\{x_1,x_2,x_3,x_5\},\{a_1(\text{Normal}),a_1(\text{Slightly high}),a_2(\text{No}),a_2(\text{Serious})\}), \\ (\{x_1,x_2,x_4,x_5\},\{a_1(\text{Normal}),a_1(\text{High}),a_2(\text{A little}),a_2(\text{Serious})\}), \\ (\{x_1,x_3,x_4,x_5\},\{a_1(\text{Normal}),a_1(\text{High}),a_2(\text{No}),a_2(\text{Serious})\}), \\ (\{x_2,x_3,x_4,x_5\},\{a_1(\text{Slightly high}),a_1(\text{High}),a_2(\text{A little}),a_2(\text{Serious})\}) \end{array} \right\},$$

$$\underline{\mathfrak{B}}_W(U,D,J) = \{(U,\emptyset),(\{x_1,x_2,x_3\},\{d(\text{No})\}),(\{x_4,x_5\},\{d(\text{Yes})\}),(\emptyset,D)\},$$

where a_1, a_2 and d denote Temperature, Headache and Flue, respectively.

Moreover, combining $\underline{\mathfrak{B}}_O(U,A,I)$ and $\underline{\mathfrak{B}}_W(U,D,J)$ with Definitions 5 and 6, we generate the following non-redundant disjunctive rules:

$$\begin{array}{ll} r_1^{\text{disj}}: & \text{Temperature(Slightly high)} \vee \text{Headache(No)} \rightarrow \text{Flu(No)}; \\ r_2^{\text{disj}}: & \text{Temperature(High)} \vee \text{Headache(Serious)} \rightarrow \text{Flu(Yes)}. \end{array}$$

Then, we know that $r_{x_1}^{\vee*}$, $r_{x_2}^{\vee*}$ and $r_{x_3}^{\vee*}$ are items of the decomposition of the disjunctive rule r_1^{disj}, and $r_{x_4}^{\vee*}$ and $r_{x_5}^{\vee*}$ are items of the decomposition of the disjunctive rule r_2^{disj}.

In addition to the combination of the truth parts of an "OR" decision rule, we continue to introduce the multi-combination of the truth parts of some "OR" decision rules below.

Let $S = (U, AT \cup \{d\})$ be a decision system and $A_1, A_2, \cdots, A_s \subseteq AT$. For any $y \in \underline{\sum_{j=1}^{s} A_j^O([x]_{\{d\}})}$, we obtain the following "OR" decision rule

$$r_y^{\vee} : \bigvee_{j=1}^{s} (\text{des}([y]_{A_j}) \rightarrow \text{des}([y]_{\{d\}})).$$

Suppose

$$r_y^{\vee*} : \left(\bigvee_{j=1}^{k_y} \text{des}([y]_{A_j}) \right) \rightarrow \text{des}([y]_{\{d\}})$$

is the combination of the truth parts of $r_y^{\vee}$. Then, we say that

$$r^{\vee*}_{\underline{\sum_{j=1}^{s} A_j^O([x]_{\{d\}})}} : \bigvee_{y \in \underline{\sum_{j=1}^{s} A_j^O([x]_{\{d\}})}} \left(\bigvee_{j=1}^{k_y} \text{des}([y]_{A_j}) \right) \rightarrow \text{des}([y]_{\{d\}})$$

is the multi-combination of the truth parts of $r_y^{\vee}\left(y\in \underline{\sum_{j=1}^{s} A_j^{O}([x]_{\{d\}})}\right)$ with their conclusions being the same.

Theorem 4. Let S be a decision system, $A_1, A_2, \cdots, A_s$ be pairwise different singleton subsets of AT with $\cup A_j = AT$, and Π be the formal decision context induced by S. Then a non-redundant disjunctive rule derived from Π is the multi-combination of the truth parts of some "OR" decision rules derived from S.

Proof. For any non-redundant disjunctive rule $B \to C$ of Π, it follows $(X, B) \in \underline{\mathfrak{B}}_O(U, A, I)$ and $(Y, C) \in \underline{\mathfrak{B}}_W(U, D, J)$, where $\underline{\mathfrak{B}}_O(U, A, I)$ is the object-oriented concept lattice of (U, A, I) and $\underline{\mathfrak{B}}_W(U, D, J)$ is Wille's concept lattice of (U, D, J). According to Table 2, without loss of generality, we assume

$$B = \{a_1(v_1^1), \cdots, a_1(v_{t_1}^1), a_2(v_1^2), \cdots, a_2(v_{t_2}^2), \cdots, a_k(v_1^k), \cdots, a_k(v_{t_k}^k)\} \text{ and } C = \{d(v_t^d)\}.$$

Then the disjunctive rule $B \to C$ can be represented as

$$a_1(v_1^1) \vee \cdots \vee a_1(v_{t_1}^1) \vee a_2(v_1^2) \vee \cdots \vee a_2(v_{t_2}^2) \vee \cdots \vee a_k(v_1^k) \vee \cdots \vee a_k(v_{t_k}^k) \to d(v_t^d). \tag{15}$$

By Definition 5, we have

$$\begin{aligned}
&B^{\diamond} \subseteq C^{\Downarrow}\\
&\Leftrightarrow \{a_1(v_1^1), \cdots, a_1(v_{t_1}^1), a_2(v_1^2), \cdots, a_2(v_{t_2}^2), \cdots, a_k(v_1^k), \cdots, a_k(v_{t_k}^k)\}^{\diamond} \subseteq \{d(v_t^d)\}^{\Downarrow}\\
&\Leftrightarrow \{a_1(v_1^1)\}^{\diamond} \cup \cdots \cup \{a_1(v_{t_1}^1)\}^{\diamond} \cup \{a_2(v_1^2)\}^{\diamond} \cup \cdots \cup \{a_2(v_{t_2}^2)\}^{\diamond} \cup \cdots \cup \{a_k(v_1^k)\}^{\diamond} \cup \cdots \cup \{a_k(v_{t_k}^k)\}^{\diamond} \subseteq \{d(v_t^d)\}^{\Downarrow}\\
&\Leftrightarrow \{a_1(v_1^1)\}^{\downarrow} \cup \cdots \cup \{a_1(v_{t_1}^1)\}^{\downarrow} \cup \{a_2(v_1^2)\}^{\downarrow} \cup \cdots \cup \{a_2(v_{t_2}^2)\}^{\downarrow} \cup \cdots \cup \{a_k(v_1^k)\}^{\downarrow} \cup \cdots \cup \{a_k(v_{t_k}^k)\}^{\downarrow} \subseteq \{d(v_t^d)\}^{\Downarrow}.
\end{aligned}$$

Thus, for any $j \in \{1, 2, \cdots, k\}$ and $i \in \{1, 2, \cdots, t_j\}$, we have $\{a_j(v_i^j)\}^{\downarrow} \subseteq \{d(v_t^d)\}^{\Downarrow}$. Moreover, based on Proposition 3, there exist $[y]_{\{a_j\}}$ with $a_j(y) = v_i^j$ and $[z]_{\{d\}}$ with $d(z) = v_t^d$ such that $[y]_{\{a_j\}} = \{a_j(v_i^j)\}^{\downarrow}$ and $[z]_{\{d\}} = \{d(v_t^d)\}^{\Downarrow}$. Thus, we get $[y]_{\{a_j\}} \subseteq [z]_{\{d\}}$. Note that $[y]_{\{a_j\}} \subseteq [z]_{\{d\}}$ can be represented as $[x]_{\{a_j\}} \subseteq [x]_{\{d\}}$, where $x \in [y]_{\{a_j\}} \cap [z]_{\{d\}}$. Besides, considering that $A_1, A_2, \cdots, A_s$ are pairwise different singleton subsets of AT with $\cup A_j = AT$, we suppose $A_1 = \{a_1\}, A_2 = \{a_2\}, \cdots, A_s = \{a_s\}$. So, $\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})$ is a truth part of the "OR" decision rule

$$r_x^{\vee} : \bigvee_{j=1}^{s} (\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}})).$$

Moreover, for any truth part $\mathrm{des}([x]_{A_l}) \to \mathrm{des}([x]_{\{d\}})$ of $r_x^{\vee}$, we have $[x]_{\{a_l\}} \subseteq [x]_{\{d\}}$, yielding $\{a_l(v_i^l)\}^{\downarrow} \subseteq \{d(v_t^d)\}^{\Downarrow}$. As a result, we obtain $\{a_l(v_i^l)\}^{\diamond} \subseteq \{d(v_t^d)\}^{\Downarrow}$, which means that $\{a_l(v_i^l)\}^{\diamond\Box} \to d(v_t^d)$ is a disjunctive rule. Since $B \to C$ is non-redundant, we get $\{a_l(v_i^l)\}^{\diamond\Box} \subseteq B$ due to $C = \{d(v_t^d)\}$, yielding $a_l(v_i^l) \in B$. That is, $\mathrm{des}([x]_{A_l}) \to \mathrm{des}([x]_{\{d\}})$ is an item of the decomposition of the disjunctive rule $B \to C$.

To sum up, each $a_j(v_i^j) \to d(v_t^d)$ with $j \in \{1, 2, \cdots, k\}$ and $i \in \{1, 2, \cdots, t_j\}$ is a truth part of an "OR" decision rule whose truth parts are items of the decomposition of the disjunctive rule $B \to C$. Consequently, by Eq. (15), we conclude that the disjunctive rule $B \to C$ is the multi-combination of the truth parts of some "OR" decision rules. □

From the proof of Theorem 4, a non-redundant disjunctive rule $B \to C$ derived from Π is in fact the multi-combination of the truth parts of the "OR" decision rules (derived from S) with their conclusions being the same as that of $B \to C$.

Example 5. Continued with Example 4. We find that the non-redundant disjunctive rule r_1^{disj} is the multi-combination of the truth parts of the "OR" decision rules $r_{x_1}^{\vee}$, $r_{x_2}^{\vee}$ and $r_{x_3}^{\vee}$. Moreover, it is easy to see that the conclusions of $r_{x_1}^{\vee}$, $r_{x_2}^{\vee}$ and $r_{x_3}^{\vee}$ are the same. Similarly, the non-redundant disjunctive rule r_2^{disj} is the multi-combination of the truth parts of the "OR" decision rules $r_{x_4}^{\vee}$ and $r_{x_5}^{\vee}$ with their conclusions being the same.

4.2. Support and certainty factors for rules in multigranulation rough sets and concept lattices

As is well known, how to evaluate the decision performance of a rule has become a very important issue in both multigranulation rough sets and concept lattices [29, 48, 70]. By considering different requirements of decision-making in the real world, a variety of measurements have been presented in recent years. In what follows, we only discuss support and certainty factors for rules in multigranulation rough sets which are called support and confidence for rules in concept lattices, respectively.

Definition 7 *[48, 70].* Let S be a decision system, $A_1, A_2, \cdots, A_s \subseteq AT$ and $x \in U$. Then,

(i) the support factor of an "AND" decision rule $\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})$ is defined as

$$\text{Supp}\left(\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})\right) = \min\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|U|} : j = 1, 2, \cdots, s\right\}; \tag{16}$$

(ii) the support factor of an "OR" decision rule $\bigvee_{j=1}^{s}(\text{des}([x]_{A_j}) \to \text{des}([x]_{\{d\}}))$ is defined as

$$\text{Supp}\left(\bigvee_{j=1}^{s}(\text{des}([x]_{A_j}) \to \text{des}([x]_{\{d\}}))\right) = \max\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|U|} : j = 1, 2, \cdots, s\right\}; \tag{17}$$

(iii) the certainty factor of an "AND" decision rule $\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})$ is defined as

$$\text{Cer}\left(\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})\right) = \min\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|[x]_{A_j}|} : j = 1, 2, \cdots, s\right\}; \tag{18}$$

(iv) the certainty factor of an "OR" decision rule $\bigvee_{j=1}^{s}(\text{des}([x]_{A_j}) \to \text{des}([x]_{\{d\}}))$ is defined as

$$\text{Cer}\left(\bigvee_{j=1}^{s}(\text{des}([x]_{A_j}) \to \text{des}([x]_{\{d\}}))\right) = \max\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|[x]_{A_j}|} : j = 1, 2, \cdots, s\right\}. \tag{19}$$

Definition 8 *[3, 65].* Let Π be a formal decision context, $x \in U$ and $x^{\uparrow} \to x^{\Uparrow}$ be a granular rule. Then the support and confidence of $x^{\uparrow} \to x^{\Uparrow}$ are respectively defined as

$$\text{Supp}(x^{\uparrow} \to x^{\Uparrow}) = \frac{|x^{\uparrow\downarrow} \cap x^{\Uparrow\Downarrow}|}{|U|} \tag{20}$$

and

$$\text{Conf}(x^{\uparrow} \to x^{\Uparrow}) = \frac{|x^{\uparrow\downarrow} \cap x^{\Uparrow\Downarrow}|}{|x^{\uparrow\downarrow}|}. \tag{21}$$

Definition 9 *[51].* Let Π be a formal decision context, $\underline{\mathfrak{B}}_O(U, A, I)$ be the object-oriented concept lattice of (U, A, I) and $\underline{\mathfrak{B}}_W(U, D, J)$ be Wille's concept lattice of (U, D, J). For any disjunctive rule $B \to C$, the support and confidence of $B \to C$ are respectively defined as

$$\text{Supp}(B \to C) = \frac{|B^{\diamond} \cap C^{\Downarrow}|}{|U|} \tag{22}$$

and

$$\text{Conf}(B \to C) = \frac{|B^{\diamond} \cap C^{\Downarrow}|}{|B^{\diamond}|}. \tag{23}$$

Theorem 5. Let S be a decision system, $A_1, A_2, \cdots, A_s$ be pairwise different subsets of AT with $\cup A_j = AT$, and Π be the induced formal decision context. For any $x \in \underline{\sum_{j=1}^{s} A_j^P([x]_{\{d\}})}$, $\text{Supp}\left(\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})\right) \geq \text{Supp}(x^{\uparrow} \to x^{\Uparrow})$ and $\text{Cer}\left(\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})\right) = \text{Conf}(x^{\uparrow} \to x^{\Uparrow})$.

Proof. Since $x \in \underline{\sum_{j=1}^{s} A_j}^P([x]_{\{d\}})$, we obtain

$$[x]_{A_j} \subseteq [x]_{\{d\}} \text{ for all } 1 \leq j \leq s. \tag{24}$$

By Eq. (16), we have

$$\mathrm{Supp}\left(\left(\bigwedge_{j=1}^{s} \mathrm{des}([x]_{A_j})\right) \to \mathrm{des}([x]_{\{d\}})\right) = \min\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|U|} : j = 1, 2, \cdots, s\right\} = \min\left\{\frac{|[x]_{A_j}|}{|U|} : j = 1, 2, \cdots, s\right\}.$$

According to Eq. (11), we get $x^{\uparrow\downarrow} = [x]_{A_1} \cap [x]_{A_2} \cap \cdots \cap [x]_{A_s}$ and $x^{\Uparrow\Downarrow} = [x]_{\{d\}}$. Then, it follows from Eqs. (20) and (24) that

$$\begin{aligned}
\mathrm{Supp}(x^{\uparrow} \to x^{\Uparrow}) &= \frac{|x^{\uparrow\downarrow} \cap x^{\Uparrow\Downarrow}|}{|U|} \\
&= \frac{|\left([x]_{A_1} \cap [x]_{A_2} \cap \cdots \cap [x]_{A_s}\right) \cap [x]_{\{d\}}|}{|U|} \\
&= \frac{|[x]_{A_1} \cap [x]_{A_2} \cap \cdots \cap [x]_{A_s}|}{|U|} \\
&\leq \min\left\{\frac{|[x]_{A_j}|}{|U|} : j = 1, 2, \cdots, s\right\} \\
&= \mathrm{Supp}\left(\left(\bigwedge_{j=1}^{s} \mathrm{des}([x]_{A_j})\right) \to \mathrm{des}([x]_{\{d\}})\right).
\end{aligned}$$

As a result, $\mathrm{Supp}\left(\left(\bigwedge_{j=1}^{s} \mathrm{des}([x]_{A_j})\right) \to \mathrm{des}([x]_{\{d\}})\right) \geq \mathrm{Supp}(x^{\uparrow} \to x^{\Uparrow})$ is at hand.

Moreover, based on Eqs. (18), (21) and (24), we obtain

$$\mathrm{Cer}\left(\left(\bigwedge_{j=1}^{s} \mathrm{des}([x]_{A_j})\right) \to \mathrm{des}([x]_{\{d\}})\right) = \min\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|[x]_{A_j}|} : j = 1, 2, \cdots, s\right\} = 1$$

and

$$\mathrm{Conf}(x^{\uparrow} \to x^{\Uparrow}) = \frac{|x^{\uparrow\downarrow} \cap x^{\Uparrow\Downarrow}|}{|x^{\uparrow\downarrow}|} = 1.$$

That is, $\mathrm{Cer}\left(\left(\bigwedge_{j=1}^{s} \mathrm{des}([x]_{A_j})\right) \to \mathrm{des}([x]_{\{d\}})\right) = \mathrm{Conf}(x^{\uparrow} \to x^{\Uparrow})$. □

Combining Theorem 1 with Theorem 5, we find that the same rule is defined with a same certainty factor but a different support factor in pessimistic multigranulation rough sets and concept lattices. This can also be confirmed by the following example.

Example 6. Continued with Example 2 in which $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$. Note that $x_3 \in \underline{A_1 + A_2}^P([x_3]_{\{d\}})$. Then, based on the results obtained in Example 2, we have

$$\mathrm{Supp}\left(\mathrm{des}([x_3]_{A_1}) \wedge \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}})\right) = \min\left\{\frac{|[x_3]_{A_1} \cap [x_3]_{\{d\}}|}{|U|}, \frac{|[x_3]_{A_2} \cap [x_3]_{\{d\}}|}{|U|}\right\} = \min\left\{\frac{2}{5}, \frac{2}{5}\right\} = \frac{2}{5},$$

$$\mathrm{Supp}(x_3^{\uparrow} \to x_3^{\Uparrow}) = \frac{|x_3^{\uparrow\downarrow} \cap x_3^{\Uparrow\Downarrow}|}{|U|} = \frac{|\{x_3\} \cap \{x_1, x_2, x_3\}|}{|U|} = \frac{1}{5},$$

$$\mathrm{Cer}\left(\mathrm{des}([x_3]_{A_1}) \wedge \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}})\right) = \min\left\{\frac{|[x_3]_{A_1} \cap [x_3]_{\{d\}}|}{|[x_3]_{A_1}|}, \frac{|[x_3]_{A_2} \cap [x_3]_{\{d\}}|}{|[x_3]_{A_2}|}\right\} = 1,$$

$$\mathrm{Conf}(x_3^{\uparrow} \to x_3^{\Uparrow}) = \frac{|x_3^{\uparrow\downarrow} \cap x_3^{\Uparrow\Downarrow}|}{|x_3^{\uparrow\downarrow}|} = \frac{|\{x_3\} \cap \{x_1, x_2, x_3\}|}{|\{x_3\}|} = 1.$$

Thus, we obtain

$$\mathrm{Supp}\left(\mathrm{des}([x_3]_{A_1}) \wedge \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}})\right) > \mathrm{Supp}(x_3^{\uparrow} \to x_3^{\Uparrow})$$

and

$$\mathrm{Cer}\left(\mathrm{des}([x_3]_{A_1}) \wedge \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}})\right) = \mathrm{Conf}(x_3^{\uparrow} \to x_3^{\Uparrow}).$$

Note that both $\mathrm{des}([x_3]_{A_1}) \wedge \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}}$ and ${x_3}^{\uparrow} \to {x_3}^{\Uparrow}$ are

$$(\mathrm{Temperature}, \mathrm{Slightly\ high}) \wedge (\mathrm{Headache}, \mathrm{No}) \to (\mathrm{Flu}, \mathrm{No}).$$

So, Example 6 confirms that the same rule is defined with a same certainty factor but a different support factor in pessimistic multigranulation rough sets and concept lattices.

Theorem 6. Let S be a decision system, $A_1, A_2, \cdots, A_s$ be pairwise different singleton subsets of AT with $\cup A_j = AT$, and Π be the induced formal decision context. If a disjunctive rule $B \to C$ with $B = \{a_1(a_1(x)), a_2(a_2(x)), \cdots, a_s(a_s(x))\}$ and $C = \{d(d(x))\}$ is the combination of the truth parts of an "OR" decision rule $\bigvee_{j=1}^{s}(\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))$, then $\mathrm{Supp}(B \to C) \geq \mathrm{Supp}\left(\bigvee_{j=1}^{s}(\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))\right)$ and $\mathrm{Conf}(B \to C) = \mathrm{Cer}\left(\bigvee_{j=1}^{s}(\mathrm{des}([x]_j) \to \mathrm{des}([x]_{\{d\}}))\right)$.

Proof. Without loss of generality, we suppose $A_1 = \{a_1\}, A_2 = \{a_2\}, \cdots, A_s = \{a_s\}$. Since the disjunctive rule $B \to C$ with $B = \{a_1(a_1(x)),\ a_2(a_2(x)),\ \cdots,\ a_s(a_s(x))\}$ and $C = \{d(d(x))\}$ is assumed to be the combination of the truth parts of the "OR" decision rule $\bigvee_{j=1}^{s}(\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))$, it follows from Eq. (22) that

$$\begin{aligned}
\mathrm{Supp}(B \to C) &= \frac{|B^{\Diamond} \cap C^{\Downarrow}|}{|U|} \\
&= \frac{|\{a_1(a_1(x)),\ a_2(a_2(x)),\ \cdots,\ a_s(a_s(x))\}^{\Diamond} \cap \{d(d(x))\}^{\Downarrow}|}{|U|} \\
&= \frac{|\left(\{a_1(a_1(x))\}^{\Diamond} \cup \{a_2(a_2(x))\}^{\Diamond} \cup \cdots \cup \{a_s(a_s(x))\}^{\Diamond}\right) \cap \{d(d(x))\}^{\Downarrow}|}{|U|} \\
&= \frac{|\left(\{a_1(a_1(x))\}^{\downarrow} \cup \{a_2(a_2(x))\}^{\downarrow} \cup \cdots \cup \{a_s(a_s(x))\}^{\downarrow}\right) \cap \{d(d(x))\}^{\Downarrow}|}{|U|} \\
&= \frac{|\left([x]_{A_1} \cup [x]_{A_2} \cup \cdots \cup [x]_{A_s}\right) \cap [x]_{\{d\}}|}{|U|} \\
&= \frac{|([x]_{A_1} \cap [x]_{\{d\}}) \cup ([x]_{A_2} \cap [x]_{\{d\}}) \cap \cdots \cap ([x]_{A_s} \cap [x]_{\{d\}})|}{|U|} \\
&\geq \max\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|U|} : j = 1, 2, \cdots, s\right\} \\
&= \mathrm{Supp}\left(\bigvee_{j=1}^{s}(\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))\right).
\end{aligned}$$

So, $\mathrm{Supp}(B \to C) \geq \mathrm{Supp}\left(\bigvee_{j=1}^{s}(\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))\right)$ is at hand.

Moreover, based on Eqs. (19) and (23), we obtain

$$\mathrm{Conf}(B \to C) = \frac{|B^{\Diamond} \cap C^{\Downarrow}|}{|B^{\Diamond}|} = 1$$

and

$$\mathrm{Cer}\left(\bigvee_{j=1}^{s}(\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))\right) = \max\left\{\frac{|[x]_{A_j} \cap [x]_{\{d\}}|}{|[x]_{A_j}|} : j = 1, 2, \cdots, s\right\} = 1.$$

Consequently, it follows $\mathrm{Conf}(B \to C) = \mathrm{Cer}\left(\bigvee_{j=1}^{s}(\mathrm{des}([x]_{A_j}) \to \mathrm{des}([x]_{\{d\}}))\right)$. □

Combining Theorem 4 with Theorem 6, we find that the same rule is defined with a same certainty factor but a different support factor in optimistic multigranulation rough sets and concept lattices. This can be confirmed by the following example.

Example 7. Continued with Example 4. Consider the following disjunctive rule $B \to C$:

$$r_1^{\mathrm{disj}} : \text{ Temperature(Slightly high)} \vee \text{Headache(No)} \to \text{Flu(No)}.$$

It is easy to check that $B \to C$ is the combination of the truth parts of the following "OR" decision rule

$$r_{x_3}^{\vee} : \text{ Temperature(Slightly high)} \to \text{Flu(No)} \;\; \vee \;\; \text{Headache(No)} \to \text{Flu(No)}.$$

Let $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$. Then, we have

$$\mathrm{Supp}\,(\mathrm{des}([x_3]_{A_1}) \vee \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}})) = \max\left\{\frac{|[x_3]_{A_1} \cap [x_3]_{\{d\}}|}{|U|}, \frac{|[x_3]_{A_2} \cap [x_3]_{\{d\}}|}{|U|}\right\} = \max\left\{\frac{2}{5}, \frac{2}{5}\right\} = \frac{2}{5},$$

$$\mathrm{Supp}(B^{\Diamond} \to C^{\Downarrow}) = \frac{|B^{\Diamond} \cap C^{\Downarrow}|}{|U|} = \frac{|\{x_1, x_2, x_3\} \cap \{x_1, x_2, x_3\}|}{|U|} = \frac{3}{5},$$

$$\mathrm{Cer}\,(\mathrm{des}([x_3]_{A_1}) \vee \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}})) = \max\left\{\frac{|[x_3]_{A_1} \cap [x_3]_{\{d\}}|}{|[x_3]_{A_1}|}, \frac{|[x_3]_{A_2} \cap [x_3]_{\{d\}}|}{|[x_3]_{A_2}|}\right\} = 1,$$

$$\mathrm{Conf}(B^{\Diamond} \to C^{\Downarrow}) = \frac{|B^{\Diamond} \cap C^{\Downarrow}|}{|B^{\Diamond}|} = \frac{|\{x_1, x_2, x_3\} \cap \{x_1, x_2, x_3\}|}{|\{x_1, x_2, x_3\}|} = 1.$$

Thus, we obtain

$$\mathrm{Supp}(B^{\Diamond} \to C^{\Downarrow}) > \mathrm{Supp}\,(\mathrm{des}([x_3]_{A_1}) \vee \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}}))$$

and

$$\mathrm{Conf}(B^{\Diamond} \to C^{\Downarrow}) = \mathrm{Cer}\,(\mathrm{des}([x_3]_{A_1}) \vee \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}}))\,.$$

Moreover, it is easy to observe that $B^{\Diamond} \to C^{\Downarrow}$ and $\mathrm{des}([x_3]_{A_1}) \vee \mathrm{des}([x_3]_{A_2}) \to \mathrm{des}([x_3]_{\{d\}})$ are the same. To sum up, Example 7 confirms that the same rule is defined with a same certainty factor but a different support factor in optimistic multigranulation rough sets and concept lattices.

4.3. Algorithm complexity analysis of rule acquisition in multigranulation rough sets and concept lattices

Let $S = (U, AT \cup \{d\})$ be a decision system, $A_1, A_2, \cdots, A_s \subseteq AT$, $\Pi = (U, A, I, D, J)$ be the formal decision context induced by S, $\underline{\mathfrak{B}}_O(U, A, I)$ be the object-oriented concept lattice of (U, A, I), and $\underline{\mathfrak{B}}_W(U, D, J)$ be Wille's concept lattice of (U, D, J).

Then the procedures of computing "AND" decision rules, granular rules, "OR" decision rules and non-redundant disjunctive rules can respectively be depicted by Algorithms 1, 2, 3 and 4. Their time complexities are $O(s|AT||U|^2)$, $O((|A| + |D|)|U|^2)$, $O(s|AT||U|^2)$ and $O((|L_O| + |L_W|)|L_O||L_W||U|)$, respectively. Here, $|L_O|$ denotes the cardinality of $\underline{\mathfrak{B}}_O(U, A, I)$ and $|L_W|$ denotes that of $\underline{\mathfrak{B}}_W(U, D, J)$. Moreover, it is easy to observe that the time complexities of Algorithms 1, 2 and 3 are all polynomial while that of Algorithm 4 is exponential.

Algorithm 1 Computing "AND" decision rules from a decision system

Require: A decision system $S = (U, AT \cup \{d\})$ with $A_1, A_2, \cdots, A_s \subseteq AT$.
Ensure: "AND" decision rules of S.

Initialize $\Omega = \emptyset$;
Build the partitions $U/IND(A_j) = \{[x]_{A_j} : x \in U\}$ $(j = 1, 2, \cdots, s)$ and $U/IND(\{d\}) = \{[x]_{\{d\}} : x \in U\}$.
For each $[x]_{\{d\}}$
    Compute the pessimistic multigranulation lower approximation $\underline{\sum_{j=1}^{s} A_j^P}([x]_{\{d\}})$;
    If $\underline{\sum_{j=1}^{s} A_j^P}([x]_{\{d\}}) \neq \emptyset$
        For each $x \in \underline{\sum_{j=1}^{s} A_j^P}([x]_{\{d\}})$
            $\Omega \leftarrow \Omega \cup \left\{\left(\bigwedge_{j=1}^{s} \text{des}([x]_{A_j})\right) \to \text{des}([x]_{\{d\}})\right\}$;
        End For
    End If
End For
Return Ω.

Algorithm 2 Computing granular rules from a formal decision context

Require: A formal decision context $\Pi = (U, A, I, D, J)$.
Ensure: Granular rules of Π.

Initialize $\Omega = \emptyset$;
For each $x \in U$
    If $x^{\uparrow\downarrow} \subseteq x^{\Uparrow\Downarrow}$
        $\Omega \leftarrow \Omega \cup \{x^{\uparrow} \to x^{\Downarrow}\}$;
    End If
End For
Return Ω.

5. Conclusions

In this section, we draw some conclusions to show the main contributions of our paper and give an outlook for further study.

(i) A brief summary of our study

To shed some light on the comparison and combination of rough set theory, granular computing and formal concept analysis, this study has investigated the relationship between multigranulation rough sets and concept lattices from the perspectives of differences and relations between rules, support and certainty factors for rules, and algorithm complexity analysis of rule acquisition. Some interesting results have been obtained in this paper. More specifically, 1) "AND" decision rules in pessimistic multigranulation rough sets have been proved to be granular rules in concept lattices, but the inverse may not be true; 2) the combination of the truth parts of an "OR" decision rule in optimistic multigranulation rough sets has been shown to be an item of the decomposition of the disjunctive rule in concept lattices; 3) each non-redundant disjunctive rule in concept lattices has been confirmed to be the multi-combination of the truth parts of some "OR" decision rules in optimistic multigranulation rough sets; 4) it has been revealed that the same rule is defined with a same certainty factor but a different support factor in multigranulation rough sets and concept lattices.

(ii) The differences and similarities between our study and the existing ones

Algorithm 3 Computing "OR" decision rules from a decision system

Require: A decision system $S = (U, AT \cup \{d\})$ with $A_1, A_2, \cdots, A_s \subseteq AT$.
Ensure: "OR" decision rules of S.

```
Initialize Ω = ∅;
Build the partitions U/IND(A_j) = {[x]_{A_j} : x ∈ U} (j = 1, 2, ···, s) and U/IND({d}) = {[x]_{d} : x ∈ U}.
For each [x]_{d}
    Compute the optimistic multigranulation lower approximation \underline{Σ_{j=1}^{s} A_j^O}([x]_{d});
    If \underline{Σ_{j=1}^{s} A_j^O}([x]_{d}) ≠ ∅
        For each x ∈ \underline{Σ_{j=1}^{s} A_j^O}([x]_{d})
            Ω ← Ω ∪ { ⋁_{j=1}^{s} (des([x]_{A_j}) → des([x]_{d})) };
        End For
    End If
End For
Return Ω.
```

Algorithm 4 Computing non-redundant disjunctive rules from a formal decision context

Require: A formal decision context $\Pi = (U, A, I, D, J)$.
Ensure: Non-redundant disjunctive rules of Π.

```
Initialize Ω = ∅;
Construct \underline{𝔅}_O(U, A, I) and \underline{𝔅}_W(U, D, J).
For each ((X, B), (Y, C)) ∈ \underline{𝔅}_O(U, A, I) × \underline{𝔅}_W(U, D, J) with X, B, Y, C ≠ ∅
    If X ⊆ Y, there does not exist (X_0, B_0) ∈ \underline{𝔅}_O(U, A, I) such that X ⊂ X_0 ⊆ Y, and there does not exist
    (Y_0, C_0) ∈ \underline{𝔅}_W(U, D, J) such that X ⊆ Y_0 ⊂ Y
        Ω ← Ω ∪ {B → C};
    End If
End For
Return Ω.
```

In what follows, we mainly distinguish the differences between our study and the existing ones with respect to the granulation environment, research objective, angle of thinking, and characteristics of interdisciplinary studies.

- Our work is different from the ones in [22, 25, 62] as far as the granulation environment is concerned. In fact, our comparative study was made under multigranulation environment, while those in [22, 25, 62] were done under single granulation environment.

- Our research is different from the ones in [29, 33] in terms of the research objective. More specifically, the current study was to compare and combine rough set theory, granular computing and formal concept analysis through "AND" decision rules, "OR" decision rules, granular rules and disjunctive rules, while reference [29] is to reveal the relationship between decision rules and granular rules, and reference [33] is to reduce the size of objects of a formal decision context without any effect on the non-redundant decision rules.

- Our study is different from the one in [21] with regard to the angle of thinking. To be more concrete, our research transformed decision systems into formal decision contexts for comparison of different rules, while the research in [21] established a concept-lattice-based rough set model by inducing a lattice structure from an information system.

- This paper is different from the ones in [5, 58] with respect to the characteristics of interdisciplinary studies.

More detailedly, multigranulation rough sets and concept lattices were comparatively studied in this paper by rule acquisition, while covering-based rough sets and concept lattices were related to each other in [5, 58] through approximation operators and reduction.

In addition to the above differences, there are similarities between our contribution and the existing ones. For example, when rough set theory is integrated with formal concept analysis, it is necessary to establish an equivalence relation from a formal context or inversely induce a lattice structure from an information system.

(iii) An outlook for further study

Note that the existing algorithm of extracting non-redundant disjunctive rules from a formal decision context takes exponential time in the worst case. One may firstly transform the formal decision context into a decision system, and then obtain the non-redundant disjunctive rules via the multi-combination of the truth parts of the "OR" decision rules with their conclusions being the same. This may be a feasible way of reducing the time complexity of computing non-redundant disjunctive rules since deriving "OR" decision rules in optimistic multigranulation rough sets only takes polynomial time. This problem will be addressed detailedly in our future work.

Moreover, the current study was to compare and combine rough set theory, granular computing and formal concept analysis via the classical multigranulation rough sets and concept lattices. Note that both the classical multigranulation rough sets and concept lattices have been generalized and developed by some researchers [17, 31, 36, 37, 47, 60, 68]. Then it is natural to further compare and combine rough set theory, granular computing and formal concept analysis based on the generalized multigranulation rough sets and concept lattices. This issue will also be discussed in our future work.

Acknowledgements

The authors would like to thank anonymous reviewers for their valuable comments and helpful suggestions which lead to a significant improvement on the manuscript. This work was supported by the National Natural Science Foundation of China (Nos. 61305057, 61322211, 61202018 and 61203283) and the Natural Science Research Foundation of Kunming University of Science and Technology (No. 14118760).

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