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Object-oriented interval-set concept lattices

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ABSTRACT

Formal concept analysis and rough set are two kinds of efficient mathematical tools for data analysis and knowledge discovery. By combining these two theories, object-oriented and property-oriented concept lattices are proposed. Interval set theory is proposed to describe a partially-known concept by a lower bound and a upper bound. In order to obtain the more accurate extension and intension for a partially-known object-oriented concept, we introduce the theory of interval sets into the object-oriented concept lattice, and propose an object-oriented interval-set concept lattice. Properties of them are investigated. Relationships among interval-set concept lattices, object-oriented interval-set concept lattices and property-oriented interval-set concept lattices are discussed. By discussing the relationships between the object-oriented concept lattice and the object-oriented interval-set concept lattice, an approach to construct object-oriented interval-set concept lattices are established.

1. Introduction

Formal concept analysis is proposed as one of the effective mathematical tools for data analysis and knowledge discovery [30]. This theory reflects the information and knowledge hidden in the data through the formal concepts and the hierarchical structure among them. Formal concepts together with a partial relation form a complete lattice, called a concept lattice [6,30]. Formal concepts and the concept lattice are the central notions in formal concept analysis, and the corresponding Hasse diagram realizes the visualization of data. It has been widely applied to many fields such as expert system, data mining, information search, knowledge engineering and software engineering [1,5,11–13,16,17,27,29] in recent years.

Rough set [22–24], proposed by Pawlak, and formal concept analysis are two different tools of analyzing data and dealing with uncertainty. Many scholars pay much attention to the study of these two theories and provide new methods for data analysis. Dubois [2], Düntsch [3,4], Kent [9], Saquer [25] and Yao [32–35] discuss four kinds of operators in data analysis by comparing concept lattices and rough sets, and propose dual concept lattices, object-oriented concept lattices and property-oriented concept lattices. Ma and Zhang [19] discuss the axiomatic characterizations of dual concept lattices. The approach to acquire object-oriented concept lattices by dividing the power set of attributes into layered sets is investigated in [21]. Shen et al. [26] show the relationship between contexts, closure spaces, and complete lattices according to the concept lattice functors. Tan et al. [28] study the connections between covering-based rough sets and concept lattices. Guo et al. [7] examine the categorical representation of algebraic domains by using rough approximate concepts. Hu et

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al. [8] introduce the approximation concepts in concept lattices. Block relations are also established for a fuzzy formal context in [10]. Furthermore, Li et al. depict the approximate concept and rule acquisition in incomplete decision contexts [15], investigate the rule acquisition and rule-preserved compression [16,17] and compare the reductions for the decision contexts [18].

For an incomplete or uncertain data, it is difficult to obtain completely known extensions or intensions of all concepts. In order to describe the partially-known concepts more clearly, interval sets are defined by Yao [31,36] to display the possible ranges of the extension and the intension for a partially-known concept. An interval set is a closed interval with the endpoints being two subsets, called the lower and upper bounds, respectively. The family of subsets between the lower and upper bounds constructs an interval set. Objects in the lower bound must be the elements of the real extension of the partially-known concept, and objects in the complement of the upper bound are not the elements of the extension of the concept. There exist many ranges to describe the extension and the intension of a partially-known concept. Which one is the best? How can we make the range more accurate to denote the concept? Since the extension and the intension of a formal concept determine each other uniquely, in order to depict the partially-known concept more accurately, Ma et al. [20] introduce the interval set into formal concept analysis, and define a pair of sufficient operators between the interval-set power sets of objects and attributes. Then interval-set concepts and interval-set concept lattices are constructed to get the ranges of the extension and intension of a partially-known concept. Possible concepts related to the partially-known concept are then obtained. Yao [37] discusses the interval sets and three-way concept analysis for incomplete contexts. And Li et al. [14] investigate an interval set model for learning rules for an incomplete information table. For a partially-known object-oriented (OO, for short) concept, on the basis of the ideas of interval sets, we can show the possible ranges of the OO extension and OO intension. In order to make the lower and upper bounds more clearly, we introduce the object-oriented interval-set concept lattice (OOIS concept lattice, for short). Based on the relationships between the concept lattice and the object-oriented concept lattice, it is possible to discuss the relationships between interval-set concept lattices and object-oriented interval-set concept lattices. According to the definitions of the sufficiency operators, the necessity operators and the possibility operators, the relationships among the interval-set concept lattice, the object-oriented interval-set concept lattice and the property-oriented interval-set concept lattice are worth to discussed.

In this paper, we introduce the notion of an interval set into the object-oriented concept lattice, and construct an object-oriented interval-set concept lattice. We first propose two pairs of operators between the interval-set power sets of objects and attributes. Then the object-oriented interval-set concept and a partial relation on them are depicted. The object-oriented interval-set concept lattice is then obtained. Related properties are shown. Relationships among the interval-set concept lattice, the object-oriented interval-set concept lattice and the property-oriented interval-set concept lattice are studied. By studying the relationships between object-oriented concept lattices and object-oriented interval-set concept lattices, we show an approach to construct object-oriented interval-set concept lattice by using the object-oriented concept lattice.

In Section 2, we first review some basic notions and results about interval sets, concept lattices and interval-set concept lattices. In Section 3, we propose the object-oriented interval-set concept, and construct object-oriented interval-set concept lattice. Relationships among the interval-set concept lattice, the object-oriented interval-set concept lattice and the property-oriented interval-set concept lattice are discussed. In Section 4, by dividing the object-oriented interval-set concept lattice into four parts, and the relationships among them are investigated. Applying the relationships, an approach to construct the object-oriented interval-set concept lattice are proposed. We then conclude the paper with a summary in Section 5.

2. Interval-set concept lattices

In this section, basic notions and properties of interval sets, concept lattices and interval-set concept lattices are introduced [6,20,30,31,36].

2.1. Interval sets and its operations

An interval set is a family of subsets of a universe, which is used to denote a range of the extension for a partially-known concept by using a lower bound and an upper bound [31,36]. It is introduced as follows:

Let U be a universe of discourse, and 2^U the power set of U. For the power set lattice $(2^U, \subseteq)$, a closed interval set $\mathcal{X}$ is defined by

$$\mathcal{X} = [X_l, X_u] = \{X \in 2^U \mid X_l \subseteq X \subseteq X_u,\ X_l,\ X_u \subseteq U\} \tag{2.1}$$

with $X_l \subseteq X_u$. X_l and X_u show the lower bound and the upper bound, respectively, of the interval set $\mathcal{X}$.

Obviously, any interval set $\mathcal{X}$ is a subset of 2^U, i.e. $\mathcal{X} \subseteq 2^U$. The set of all interval sets of U is denoted by

$$\mathcal{I}(2^U) = \{\mathcal{X} = [X_l, X_u] \mid X_l, X_u \subseteq U, X_l \subseteq X_u\},$$

called the interval-set power set of U.

For any subset $X \in 2^U$, let

$$\widehat{X} = [X, X].$$

$\widehat{X}$ is also an interval set with $X_l = X_u = X$, called a degenerated interval set, or a single interval set. Then the single interval set $\widehat{X}$ is actually a single set with one set X, i.e. $\widehat{X} = \{X\}$. Furthermore, the two sets

$$\widehat{\emptyset_U} = [\emptyset_U, \emptyset_U], \ \widehat{U} = [U, U]$$

are also interval sets in $\mathcal{I}(2^U)$.

Corresponding to the set operations of the intersection $\cap$, the union $\cup$, the difference $-$ and the complement ^{c} on 2^U, Yao introduces the similar operations on $\mathcal{I}(2^U)$ [31,36]: for any two interval sets $\mathcal{X} = [X_l, X_u]$, $\mathcal{Y} = [Y_l, Y_u] \in \mathcal{I}(2^U)$,

$$\begin{aligned} &\mathcal{X} \sqcap \mathcal{Y} = [X_l \cap Y_l, X_u \cap Y_u] = \{X \cap Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}, \\ &\mathcal{X} \sqcup \mathcal{Y} = [X_l \cup Y_l, X_u \cup Y_u] = \{X \cup Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}, \\ &\mathcal{X} - \mathcal{Y} = [X_l - Y_u, X_u - Y_l] = \{X - Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}, \\ &\quad \neg\mathcal{X} = [U, U] - [X_l, X_u] = [X_u^c, X_l^c]. \end{aligned} \tag{2.2}$$

Obviously,

$$\neg\, \widehat{\emptyset_U} = \widehat{U}, \ \neg\, \widehat{U} = \widehat{\emptyset_U}, \ \text{ and } \ \widehat{X^c} = \neg\, \widehat{X}$$

for any subset $X \in \mathcal{P}(U)$.

For the interval-set power set $\mathcal{I}(2^U)$, a binary relation $\sqsubseteq$ is defined as follows: for any $\mathcal{X} = [X_l, X_u]$, $\mathcal{Y} = [Y_l, Y_u] \in \mathcal{I}(2^U)$,

$$[X_l, X_u] \sqsubseteq [Y_l, Y_u] \Leftrightarrow X_l \subseteq Y_l, \ X_u \subseteq Y_u. \tag{2.3}$$

Then $\sqsubseteq$ is a partial relation on $\mathcal{I}(2^U)$, and $(\mathcal{I}(2^U), \sqsubseteq)$ is a partial set. Applying Eq. (2.3) we can get that, for any interval sets $\mathcal{X} = [X_l, X_u]$, $\mathcal{Y} = [Y_l, Y_u] \in \mathcal{I}(2^U)$,

$$\mathcal{X} = \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y} \text{ and } \mathcal{Y} \sqsubseteq \mathcal{X} \Leftrightarrow X_l = Y_l, \ X_u = Y_u. \tag{2.4}$$

The partial relation $\sqsubseteq$ satisfies the following properties [31]: for any $\mathcal{X}, \ \mathcal{Y}, \ \mathcal{X}_1, \ \mathcal{Y}_1 \in \mathcal{I}(2^U)$,
(1) $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \sqcap \mathcal{Y} = \mathcal{X} \Leftrightarrow \mathcal{X} \sqcup \mathcal{Y} = \mathcal{Y}$;
(2) $\mathcal{X} \sqsubseteq \mathcal{Y}, \ \mathcal{X}_1 \sqsubseteq \mathcal{Y}_1 \Rightarrow \mathcal{X} \sqcap \mathcal{X}_1 \sqsubseteq \mathcal{Y} \sqcap \mathcal{Y}_1, \ \mathcal{X} \sqcup \mathcal{X}_1 \sqsubseteq \mathcal{Y} \sqcup \mathcal{Y}_1$,
(3) $\mathcal{X} \sqcap \mathcal{Y} \sqsubseteq \mathcal{X}, \ \mathcal{X} \sqcap \mathcal{Y} \sqsubseteq \mathcal{Y}, \ \mathcal{X} \sqsubseteq \mathcal{X} \sqcup \mathcal{Y}, \ \mathcal{Y} \sqsubseteq \mathcal{X} \sqcup \mathcal{Y}$.

Remark 2.1. For any subsets $X, \ Y \subseteq U$, $X \cap (\sim X) = \emptyset_U$, $X \cup (\sim X) = U$, $X - X = \emptyset_U$, and $X \subseteq Y$ implies that $X - Y = \emptyset_U$. However, these results may not be true for interval sets. Actually, for any interval sets $\mathcal{X} = [X_l, X_u]$, $\mathcal{Y} = [Y_l, Y_u] \in \mathcal{I}(2^U)$,

$$\begin{aligned} &\mathcal{X} \sqcap \neg\mathcal{X} = [\emptyset_U, X_u - X_l], \ \mathcal{X} \sqcup \neg\mathcal{X} = [X_l \cup X_u^c, X_u \cup X_l^c], \\ &\mathcal{X} - \mathcal{X} = [\emptyset_U, X_u - X_l], \\ &\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} - \mathcal{Y} = [\emptyset_U, X_u - Y_l]. \end{aligned}$$

2.2. Interval-set concept lattices

A formal context is a triplet (U, A, I), where $U = \{x_1, x_2, \cdots, x_n\}$ is a non-empty finite set of objects called a universe of discourse, $A = \{a_1, a_2, \cdots, a_m\}$ is a non-empty finite set of properties or attributes, and $I \subseteq U \times A$ is a binary relation, which is used to describe the relationships between objects and attributes. For any $x \in U$ and $a \in A$, $(x, a) \in I$, also written as xIa, means that the object x has the attribute a, or the attribute a is possessed by the object x. The complement $I^c \subseteq U \times A$ of the binary relation I satisfies: $(x, a) \in I^c \Leftrightarrow (x, a) \notin I$. Then the triple (U, A, I^c) is also a formal context, called the complement formal context of (U, A, I). If we denote $(x, a) \in I$ by 1 and $(x, a) \notin I$ by 0, a formal context and its complement formal context can be represented by a table with 0 and 1 [6,30].

For a formal context (U, A, I), 2^U and 2^A denote the power sets of U and A, respectively. For any $X \in 2^U$ and $B \in 2^A$, a pair of operators $^\uparrow : 2^U \to 2^A$ and $^\downarrow : 2^A \to 2^U$, called sufficiency operators [6,30], is defined by

$$\begin{aligned} X^\uparrow &= \{a \in A \mid \ (x, a) \in I, \ \forall x \in X\}, \\ B^\downarrow &= \{x \in U \mid \ (x, a) \in I, \ \forall a \in B\}. \end{aligned} \tag{2.5}$$

$X^\uparrow$ is the set of attributes shared by all objects in X, and $B^\downarrow$ is the set of objects which possess all attributes in B.

For any $x \in U$ and $a \in A$, by Eq. (2.5), we have

$$\{x\}^\uparrow = \{a \in A \mid \ (x, a) \in I\}, \ \{a\}^\downarrow = \{x \in U \mid \ (x, a) \in I\}. \tag{2.6}$$

For simplicity, we use $x^\uparrow$ and $a^\downarrow$ to denote the sets $\{x\}^\uparrow$ and $\{a\}^\downarrow$, respectively. Therefore,

$$(x, a) \in I \Leftrightarrow x \in a^\downarrow \Leftrightarrow a \in x^\uparrow. \tag{2.7}$$

Table 2.1
The formal context (U, A, I) of Example 2.1.

U	a	b	c	d	e
1	1	0	1	1	1
2	1	0	1	0	0
3	0	1	0	0	1
4	0	1	0	0	1
5	1	0	0	0	0
6	1	1	0	0	1

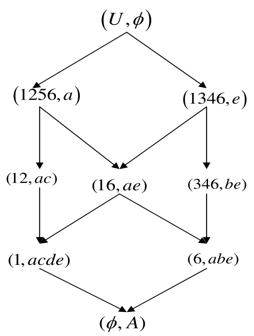

Fig. 1. Concept lattice $L(U, A, I)$.

A formal context (U, A, I) is regular if for any $x \in U$ and $a \in A$, $x^{\uparrow} \neq \emptyset_A$, $x^{\uparrow} \neq A$, $a^{\downarrow} \neq \emptyset_U$ and $a^{\downarrow} \neq U$. In this paper, the formal context (U, A, I) we discussed is always regular. According to Eqs. (2.5), (2.6) and (2.7), one can obtain that:

$$\begin{aligned} X^{\uparrow} &= \{ a \in A \mid \forall x \in U\, [\, x \in X \Rightarrow x \in a^{\downarrow}] \} \\ &= \{ a \in A \mid X \subseteq a^{\downarrow} \}, \\ B^{\downarrow} &= \{ x \in U \mid \forall a \in A\, [\, a \in A \Rightarrow a \in x^{\uparrow}] \} \\ &= \{ x \in U \mid B \subseteq x^{\uparrow} \}. \end{aligned} \tag{2.8}$$

The sufficiency operators have the following properties [6,30]: for any $X, X_1, X_2 \in 2^U$ and $B, B_1, B_2 \in 2^A$,
(1) $X_1 \subseteq X_2 \Rightarrow X_2^{\uparrow} \subseteq X_1^{\uparrow}$, $B_1 \subseteq B_2 \Rightarrow B_2^{\downarrow} \subseteq B_1^{\downarrow}$;
(2) $X \subseteq X^{\uparrow\downarrow}$, $B \subseteq B^{\downarrow\uparrow}$;
(3) $X^{\uparrow} = X^{\uparrow\downarrow\uparrow}$, $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$;
(4) $(X_1 \cup X_2)^{\uparrow} = X_1^{\uparrow} \cap X_2^{\uparrow}$, $(B_1 \cup B_2)^{\downarrow} = B_1^{\downarrow} \cap B_2^{\downarrow}$;
(5) $X \subseteq B^{\downarrow} \Leftrightarrow B \subseteq X^{\uparrow}$.

A pair (X, B) with $X \subseteq U$ and $B \subseteq A$ is called a formal concept (for short, a concept), if $X^{\uparrow} = B$ and $B^{\downarrow} = X$. X is called the extension and B the intension of the concept (X, B). The set of all concepts of the formal context (U, A, I), denoted by $L(U, A, I)$, forms a complete lattice, called a concept lattice [6,30], where the partial order $\leq$ is defined as follows: for any $(X_1, B_1), (X_2, B_2) \in L(U, A, I)$,

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow B_2 \subseteq B_1.$$

And the meet and join are given by:

$$\begin{aligned} (X_1, B_1) \wedge (X_2, B_2) &= (X_1 \cap X_2, (B_1 \cup B_2)^{\downarrow\uparrow}), \\ (X_1, B_1) \vee (X_2, B_2) &= ((X_1 \cup X_2)^{\uparrow\downarrow}, B_1 \cap B_2). \end{aligned}$$

Example 2.1. Table 2.1 depicts a formal context (U, A, I) with $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{a, b, c, d, e\}$.

In the following description, for simplicity, a set is denoted by listing its elements. For example, the set $\{1, 2, 3, 4\}$ is denoted by 1234. Fig. 1 shows the concept lattice $L(U, A, I)$ of the formal context (U, A, I) given in Table 2.1.

For an uncertain concept (X, B), we cannot get the exact extension X and the intension B. Suppose the interval set $\mathcal{X} = [X_l, X_u]$ shows the possible range of X, that is, $X_l \subseteq X \subseteq X_u$. However, X_l and X_u may not be the extensions of some formal concepts in (U, A, I). How can we get the lower and upper bounds of the interval set $\mathcal{X}$? And how can we show the more accurate ranges of extension and intension for the uncertain concept (X, B)?

Definition 2.1. [20] Let (U, A, I) be a formal context. For any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$, a pair of operators (f, g) between $\mathcal{I}(2^U)$ and $\mathcal{I}(2^A)$ are defined as follows:

$$\begin{aligned} f(\mathcal{X}) &= [X_u^{\uparrow}, X_l^{\uparrow}] = \{B \in 2^A \mid X_u^{\uparrow} \subseteq B \subseteq X_l^{\uparrow}\}, \\ g(\mathcal{B}) &= [B_u^{\downarrow}, B_l^{\downarrow}] = \{X \in 2^U \mid B_u^{\downarrow} \subseteq X \subseteq B_l^{\downarrow}\}. \end{aligned} \tag{2.9}$$

$f(\mathcal{X})$ is the set of the possible intensions for some concept with the intension $X_u^{\uparrow}$ of the concept $(X_u^{\uparrow\downarrow}, X_u^{\uparrow})$ being its lower bound, and $X_l^{\uparrow}$ of the concept $(X_l^{\uparrow\downarrow}, X_l^{\uparrow})$ being its upper bound; and $g(\mathcal{B})$ shows the set of all possible extensions for some concept, where the lower bound of it is the extension $B_u^{\downarrow}$ of the concept $(B_u^{\downarrow}, B_u^{\downarrow\uparrow})$, and the upper bound of it is the extension $B_l^{\downarrow}$ of the concept $(B_l^{\downarrow}, B_l^{\downarrow\uparrow})$.

An interval set $\mathcal{X} = [X_l, X_u]$ is used to describe the possible range of the extension of a partially-known concept. Any possible extension set $X \in \mathcal{X}$ satisfies $X_l \subseteq X \subseteq X_u$. Since $X_l^{\uparrow}$ is the intension of the concept $(X_l^{\uparrow\downarrow}, X_l^{\uparrow})$ generated by the lower bound X_l, and $X_u^{\uparrow}$ is the intension of the concept $(X_u^{\uparrow\downarrow}, X_u^{\uparrow})$ constructed by the upper bound X_u, the sufficient operator $f: \mathcal{I}(2^U) \to \mathcal{I}(2^A)$ produces the possible range of the intensions for some concept with the intension $X_u^{\uparrow}$ being the lower bound and the intension $X_l^{\uparrow}$ being the upper bound. It is obvious that $X_u^{\uparrow} \subseteq X^{\uparrow} \subseteq X_l^{\uparrow}$. Similarly, the sufficient operator $g: \mathcal{I}(2^A) \to \mathcal{I}(2^U)$ discusses the set of all possible range of the extensions for some concept with the extension $B_u^{\downarrow}$ being the lower bound and the extension $B_l^{\downarrow}$ being the upper bound.

Remark 2.2. For any interval set $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$, the lower bound X_l and the upper bound X_u may not be the extensions of some concepts. Applying Definition 2.1, the interval set $f(\mathcal{X}) = [X_u^{\uparrow}, X_l^{\uparrow}]$ shows the range of the intension of another concept. The lower bound $X_u^{\uparrow}$ and the upper bound $X_l^{\uparrow}$ are actually the intensions of the formal concepts $(X_u^{\uparrow\downarrow}, X_u^{\uparrow})$ and $(X_l^{\uparrow\downarrow}, X_l^{\uparrow})$, respectively. And for any possible extension $X \in \mathcal{X}$, $X_u^{\uparrow}$ and $X_l^{\uparrow}$ are the corresponding lower and upper bounds of the intension $X^{\uparrow}$ of the possible extension X.

Example 2.2. The formal context (U, A, I) in Example 2.1 has 9 formal concepts

$$L(U, A, I) = \{(U, \emptyset_A),\ (1256, a),\ (1346, e),\ (12, ac),\ (16, ae),\ (346, be),\ (1, acde),\ (6, abe),\ (\emptyset_U, A)\}.$$

For the interval set $\mathcal{X} = [12, 12456]$, the lower bound $\{1, 2\}$ is the extension of the formal concept $(12, ac)$, but the upper bound $\{1, 2, 4, 5, 6\}$ is not the extension of any formal concept. Applying Definition 2.1 we can get that $f(\mathcal{X}) = [\emptyset_A, ac]$, where the lower bound $\emptyset_A$ is the intension of the concept $(U, \emptyset_A)$, and the upper bound $\{a, c\}$ is the intension of the concept $(12, ac)$. Since

$$\mathcal{X} = [12, 12456] = \{12, 124, 125, 126, 1245, 1246, 1256, 12456\},$$

we can get the corresponding intensions for any set in the interval set $\mathcal{X}$:

$$12^{\uparrow} = ac, \quad 124^{\uparrow} = \emptyset_A, \quad 125^{\uparrow} = a, \quad 126^{\uparrow} = 1, \quad 1245^{\uparrow} = \emptyset_A,\ 1246^{\uparrow} = \emptyset_A,\ 1256^{\uparrow} = a,\ 12456^{\uparrow} = \emptyset_A.$$

Then the sets $\emptyset_A$ and $\{b, c\}$, respectively, are also the lower and upper bounds of the intension of any possible extension set $X \in \mathcal{X}$.

Take an interval set $\mathcal{Y} = [4, 3456]$. The lower bound $\{4\}$ and the upper bound $\{3456\}$ of the interval set $\mathcal{Y}$ are not the extensions of any formal concepts in $L(U, A, I)$. According to Definition 2.1 we get that $f(\mathcal{Y}) = [\emptyset_A, be]$ with the lower bound $\emptyset_A$ being the intension of the corresponding formal concept $(U, \emptyset_A)$, and the upper bound $\{b, e\}$ being the intension of the corresponding formal concept $(346, be)$. Furthermore, for the interval set

$$\mathcal{Y} = [4, 3456] = \{4, 34, 45, 46, 345, 346, 456, 3456\}$$

we have

$$\begin{aligned} &4^{\uparrow} = be, \quad 34^{\uparrow} = be, \quad 45^{\uparrow} = \emptyset_A, \quad 46^{\uparrow} = be, \\ &345^{\uparrow} = \emptyset_A,\ 346^{\uparrow} = be,\ 456^{\uparrow} = \emptyset_A,\ 3456^{\uparrow} = \emptyset_A. \end{aligned}$$

The sets $\emptyset_A$ and $\{b, e\}$ are also the lower and upper bounds, respectively, of the intension of any possible extension set $X \in \mathcal{Y}$. That is to say, for any $Y \in \mathcal{Y}$, $\emptyset_A \subseteq Y^{\uparrow} \subseteq \{b, e\}$.

Similarly, for the interval set $\mathcal{B} = [b, bcd]$, the sets $\{b\}$ and $\{b, c, d\}$ are not the intensions of any formal concepts in $L(U, A, I)$. According to Definition 2.1, $g(\mathcal{B}) = [\emptyset_U, 346]$. The lower bound $\emptyset_U$ is the extension of the formal concept $(\emptyset_U, A)$, and the upper bound $\{3, 4, 6\}$ is the extension of the formal concept $(346, be)$. Since $\mathcal{B} = [b, bcd] = \{b, bc, bd, bcd\}$, we can get the corresponding extension sets of the elements in $\mathcal{B}$:

$$b^{\downarrow} = 346,\ bc^{\downarrow} = \emptyset_U,\ bd^{\downarrow} = \emptyset_U,\ bcd^{\downarrow} = \emptyset_U.$$

It is obvious that the extension of any element in the interval set $\mathcal{B}$ belongs to the interval set $g(\mathcal{B})$.

For any $X \subseteq U$ and $B \subseteq A$,

$$f(\widehat{X}) = [X^{\uparrow}, X^{\uparrow}] = \widehat{X^{\uparrow}}, \quad g(\widehat{B}) = [B^{\downarrow}, B^{\downarrow}] = \widehat{B^{\downarrow}}.$$

Definition 2.2. [20] Let (U, A, I) be a formal context. For any $\mathcal{X} \in \mathcal{I}(2^U)$ and $\mathcal{B} \in \mathcal{I}(2^A)$, if $f(\mathcal{X}) = \mathcal{B}$ and $g(\mathcal{B}) = \mathcal{X}$, we call $(\mathcal{X}, \mathcal{B})$ an interval-set formal concept (for short, an IS concept). $\mathcal{X}$ is called the IS extension, and $\mathcal{B}$ the IS intension of the IS concept $(\mathcal{X}, \mathcal{B})$.

For a partially-known concept, the interval set $\mathcal{X} = [X_l, X_u]$ shows the range of the extension for the partially-known concept with X_l being the lower bound and X_u being the upper bound. $f(\mathcal{X})$ shows the possible range of the intension of the concept with the intension set $X_u^{\uparrow}$ being the lower bound and the intension set $X_l^{\uparrow}$ being the upper bound.

Remark 2.3. For the partially-known concept $C = (X, B)$, $X^{\uparrow} = B$ and $B^{\downarrow} = X$ with X and B uncertain or partially-known.

(1) The interval set $\mathcal{X} = [X_l, X_u]$ shows a range of the extension X with $X_l \subseteq X \subseteq X_u$, and the interval set $\mathcal{B} = [B_l, B_u]$ displays a range of the intension B with $B_l \subseteq B \subseteq B_u$.

(2) $X_l \subseteq X \subseteq X_u$ and $B_l \subseteq B \subseteq B_u$ imply that $X_u^{\uparrow} \subseteq X^{\uparrow} \subseteq X_l^{\uparrow}$ and $B_u^{\downarrow} \subseteq B^{\downarrow} \subseteq B_l^{\downarrow}$, respectively. Together with $X^{\uparrow} = B$ and $B^{\downarrow} = X$ we get that $f(\mathcal{X}) = [X_u^{\uparrow}, X_l^{\uparrow}]$ shows a range of the intension B, and $g(\mathcal{B}) = [B_u^{\downarrow}, B_l^{\downarrow}]$ shows a range of the extension X of the concept $C = (X, B)$.

(3) According to (1) and (2) we know that, $\mathcal{B} = [B_l, B_u]$ is a range of the intension B of the partially-known concept $C = (X, B)$. Meanwhile, $f(\mathcal{X}) = [X_u^{\uparrow}, X_l^{\uparrow}]$ also shows a range of the intension B of the concept $C = (X, B)$. However,

$$[B_l, B_u] = [X_u^{\uparrow}, X_l^{\uparrow}]$$

may not be true. That is to say, $\mathcal{B} = [B_l, B_u]$ and $f(\mathcal{X}) = [X_u^{\uparrow}, X_l^{\uparrow}]$ provide two ranges of the intension B. Similarly, $\mathcal{X} = [X_l, X_u]$ and $g(\mathcal{B}) = [B_u^{\downarrow}, B_l^{\downarrow}]$ give two ranges of the extension X of the concept $C = (X, B)$. But these two ranges may not be the same one. That is,

$$[X_l, X_u] = [B_u^{\downarrow}, B_l^{\downarrow}]$$

may not hold.

It is well-known that a concept consists of an extension and an intension, which determines each other uniquely. Based on it,

$$f(\mathcal{X}) = [X_u^{\uparrow}, X_l^{\uparrow}] = [B_l, B_u] = \mathcal{B}$$

denotes that the given range of the intension B equals to the range of the intensions obtained by the given range of the extension X. Meanwhile,

$$g(\mathcal{B}) = [B_u^{\downarrow}, B_l^{\downarrow}] = [X_l, X_u] = \mathcal{X}$$

shows that the given range of the extension X equals to the range of the extensions obtained by the given range of the intension B. Combining these two equations, the two ranges of the intension B and the extension X determines each other. Then we can get the more accurate ranges for the extension X and intension B for the partially-known concept, which produces an interval-set formal concept $(\mathcal{X}, \mathcal{B})$.

Example 2.3. Considering the formal context (U, A, I) given in Example 2.1.

(1) For a partially-known concept $C = (X, B)$, the interval set $\mathcal{X} = [1, 125]$ and $\mathcal{B} = [a, ade]$ display the ranges of the extension and the intension, respectively, of the concept. The extension X of the concept must include the object 1, and may consist of the objects 2 or 5. $\{1\}$ is the lower bound, and $\{1, 2, 5\}$ is the upper bound of the extension X for the concept. The intension B of the concept must include the attribute a, and may consist of the attributes d or e. $\{a\}$ is the lower bound, and $\{a, d, e\}$ is the upper bound of the intension B for the concept. Thus,

$$\mathcal{X} = [1, 125] = \{1, 12, 15, 125\}, \quad \mathcal{B} = [a, ade] = \{a, ad, ae, ade\}.$$

Applying Definition 2.1 we can get that

$$f(\mathcal{X}) = [125^{\uparrow}, 1^{\uparrow}] = [a, acde] = \{a, ac, ad, ae, acd, ace, ade, acde\},$$
$$g(\mathcal{B}) = [ade^{\downarrow}, a^{\downarrow}] = [1, 1256] = \{1, 12, 15, 16, 125, 126, 156, 1256\}.$$

$f(\mathcal{X}) \neq \mathcal{B}$ denotes that the interval set $f(\mathcal{X})$, the range of the intensions with $X_u^{\uparrow}$ being the lower bound and $X_l^{\uparrow}$ being the upper bound, does not equal to the interval set $\mathcal{B}$, the range of the intension of the partially-known concept. And $g(\mathcal{B}) \neq \mathcal{X}$

tells us that the interval set $g(\mathcal{B})$, the range of the extension obtained by the lower bound $B_u^{\downarrow}$ and the upper bound $B_l^{\downarrow}$ is not the interval set $\mathcal{X}$ with X_l being the lower bound and X_u being the upper bound.

Furthermore, we also get that for any possible extensions in $\mathcal{X}$ and any possible intensions in $\mathcal{B}$,

$$\begin{aligned} &1^{\uparrow} = acde \notin \mathcal{B},\ 12^{\uparrow} = ac \notin \mathcal{B}, \quad 15^{\uparrow} = 125^{\uparrow} = a \in \mathcal{B}, \\ &a^{\downarrow} = 1256 \notin \mathcal{X},\ \ ae^{\downarrow} = 16 \notin \mathcal{X}, \quad ad^{\downarrow} = ade^{\downarrow} = 1 \in \mathcal{X}. \end{aligned} \tag{2.10}$$

Any set in the interval set $\mathcal{X}$ may be the extension of the partially-known concept, and $f(\mathcal{X})$ includes all intensions with $X_u^{\uparrow}$ and $X_l^{\uparrow}$ being the lower and upper bounds, respectively. Applying Eq. (2.5), we just get two formal concepts $(1, acde)$ and $(12, ac)$, where the extensions $\{1\}$ and $\{1, 2\}$ belong to $\mathcal{X}$ but the intensions $\{a, c, d, e\}$ and $\{a, c\}$ are not in $\mathcal{B}$. Similarly, $\mathcal{B}$ consists of the possible intensions of the partially-known concept, and $g(\mathcal{B})$ shows the range of extensions with the lower bound $B_u^{\downarrow}$ and the upper bound $B_l^{\downarrow}$. According to Eq. (2.5) we just obtain the formal concepts $(16, ae)$ and $(1256, a)$. It is obvious that the intensions $\{a\} \in \mathcal{B}$, $\{a, e\} \in \mathcal{B}$, but the extensions $\{1, 6\} \notin \mathcal{X}$, $\{1, 2, 5, 6\} \notin \mathcal{X}$.

That is to say, for a partially-known concept $C = (X, B)$, we choose the range $\mathcal{X}$ of the extension X and the range $\mathcal{B}$ of the intension. It is still difficult to get the more exact extension X and intension B.

(2) Now we take the interval set $\mathcal{Y} = [1, 1256]$ including the extension X and the interval set $\mathcal{C} = [a, acde]$ consisting of the intension B the partially-known concept $C = (X, B)$. Then

$$\begin{aligned} &\mathcal{Y} = [1, 1256] = \{1, 12, 15, 16, 125, 126, 156, 1256\}, \\ &\mathcal{C} = [a, acde] = \{a, ac, ad, ae, acd, ace, acde\}. \end{aligned}$$

Applying Definition 2.1 we can get that

$$\begin{aligned} &f(\mathcal{Y}) = [1256^{\uparrow}, 1^{\uparrow}] = [a, acde] = \{a, ac, ad, ae, acd, ace, ade, acde\} = \mathcal{C}, \\ &g(\mathcal{C}) = [acde^{\downarrow}, a^{\downarrow}] = [1, 1256] = \{1, 12, 15, 16, 125, 126, 156, 1256\} = \mathcal{Y}. \end{aligned}$$

Then $f(\mathcal{Y}) = \mathcal{C}$ and $g(\mathcal{C}) = \mathcal{Y}$. Meanwhile, we also obtain that for any possible extensions in $\mathcal{X}$ and any possible intensions in $\mathcal{B}$,

$$\begin{aligned} &1^{\uparrow} = acde \in \mathcal{C},\ 12^{\uparrow} = ac \in \mathcal{C},\ 15^{\uparrow} = a \in \mathcal{C},\ 16^{*} = ae \in \mathcal{C},\ 125^{\uparrow} = 126^{\uparrow} = 156^{\uparrow} = 1256^{\uparrow} = a \in \mathcal{C}, \\ &a^{\downarrow} = 1256 \in \mathcal{Y},\ ac^{\downarrow} = 12 \in \mathcal{Y},\ ad^{\downarrow} = 1 \in \mathcal{Y},\ ae^{\downarrow} = 16 \in \mathcal{Y},\ acd^{\downarrow} = ace^{\downarrow} = ade^{\downarrow} = acde^{\downarrow} = 1 \in \mathcal{Y}. \end{aligned}$$

From these we can get four formal concepts $(12, ac)$, $(16, ae)$ and $(1256, a)$ with the extensions coming from the interval set $\mathcal{Y}$ and the intensions belonging to the interval set $\mathcal{C}$. Therefore, one of the four formal concepts $(1, acde)$, $(12, ac)$, $(16, ae)$ and $(1256, a)$ must be the partially-known concept.

Property 2.1. [20] Let (U, A, I) be a formal context. For any $\mathcal{X}, \mathcal{Y} \in \mathcal{I}(2^U)$ and $\mathcal{B}, \mathcal{C} \in \mathcal{I}(2^A)$, the following properties hold:

(1) $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow f(\mathcal{Y}) \sqsubseteq f(\mathcal{X})$, $\mathcal{B} \sqsubseteq \mathcal{C} \Rightarrow g(\mathcal{C}) \sqsubseteq g(\mathcal{B})$;
(2) $\mathcal{X} \sqsubseteq gf(\mathcal{X})$, $\mathcal{B} \sqsubseteq fg(\mathcal{B})$;
(3) $f(\mathcal{X} \sqcup \mathcal{Y}) = f(\mathcal{X}) \sqcap f(\mathcal{Y})$, $g(\mathcal{B} \sqcup \mathcal{C}) = g(\mathcal{B}) \sqcap g(\mathcal{C})$;
(4) $f(\mathcal{X}) = fgf(\mathcal{X})$, $g(\mathcal{B}) = gfg(\mathcal{B})$;
(5) $\mathcal{X} \sqsubseteq g(\mathcal{B}) \Leftrightarrow \mathcal{B} \sqsubseteq f(\mathcal{X})$;
(6) $f(\mathcal{X} \sqcap \mathcal{Y}) \sqsupseteq f(\mathcal{X}) \sqcup f(\mathcal{Y})$, $g(\mathcal{B} \sqcap \mathcal{C}) \sqsupseteq g(\mathcal{B}) \sqcup g(\mathcal{C})$;
(7) $(gf(\mathcal{X}), f(\mathcal{X}))$, $(g(\mathcal{B}), fg(\mathcal{B}))$ are both IS concepts.

Proof. Applying properties of operators $(^{\uparrow}, ^{\downarrow})$, Definition 2.1 and Definition 2.2, one can get these results. □

Theorem 2.1. *Let (U, A, I) be a formal context. For any $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$,*

$$(\mathcal{X}, \mathcal{B}) \in IL(U, A, I) \Leftrightarrow (X_l, B_u),\ (X_u, B_l) \in L(U, A, I) \text{ and } (X_l, B_u) \leq (X_u, B_l).$$

Proof. Suppose $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$, $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$ and $(\mathcal{X}, \mathcal{B}) \in IL(U, A, I)$. Applying Definition 2.2 we can get that

$$f(\mathcal{X}) = [X_u^{\uparrow}, X_l^{\uparrow}] = [B_l, B_u] = \mathcal{B} \text{ and } g(\mathcal{B}) = [B_u^{\downarrow}, B_l^{\downarrow}] = [X_l, X_u] = \mathcal{X}.$$

According to Eq. (2.4) we can get that $X_u^{\uparrow} = B_l$, $X_l^{\uparrow} = B_u$ and $B_u^{\downarrow} = X_l$, $B_l^{\downarrow} = X_u$. Therefore, $X_u^{\uparrow} = B_l$ and $B_l^{\downarrow} = X_u$ imply $(X_u, B_l) \in L(U, A, I)$, and $X_l^{\uparrow} = B_u$ and $B_u^{\downarrow} = X_l$ produce $(X_l, B_u) \in L(U, A, I)$. Furthermore, the interval set $\mathcal{X} = [X_l, X_u]$ with $X_l \subseteq X_u$ implies that $(X_l, B_u) \leq (X_u, B_l)$.

Assume that for any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$, (X_l, B_u), $(X_u, B_l) \in L(U, A, I)$ and $(X_l, B_u) \leq (X_u, B_l)$. $(X_l, B_u) \leq (X_u, B_l)$ implies that $X_l \subseteq X_u$ and $B_l \subseteq B_u$. (X_l, B_u), $(X_u, B_l) \in L(U, A, I)$ shows that $X_l^{\uparrow} = B_u$, $B_u^{\downarrow} = X_l$, and $X_u^{\uparrow} = B_l$, $B_l^{\downarrow} = X_u$. Together with Definition 2.1 we can obtain that

$$f(\mathcal{X}) = f([X_l, X_u]) = [X_u^{\uparrow}, X_l^{\uparrow}] = [B_l, B_u] = \mathcal{B},$$
$$g(\mathcal{B}) = g([B_l, B_u]) = [B_u^{\downarrow}, B_l^{\downarrow}] = [X_l, X_u] = \mathcal{X}.$$

According to Definition 2.2 we know that $(\mathcal{X}, \mathcal{B})$ is an IS concept.

For a formal context (U, A, I), we denote by $IL(U, A, I)$ the set of all IS concepts. Then for any two IS concepts $(\mathcal{X}_1, \mathcal{B}_1)$, $(\mathcal{X}_2, \mathcal{B}_2) \in IL(U, A, I)$, a binary relation $\preccurlyeq$ is defined as follows:

$$(\mathcal{X}_1, \mathcal{B}_1) \preccurlyeq (\mathcal{X}_2, \mathcal{B}_2) \Leftrightarrow \mathcal{X}_1 \sqsubseteq \mathcal{X}_2 \Leftrightarrow \mathcal{B}_2 \sqsubseteq \mathcal{B}_1. \tag{2.11}$$

The binary relation $\preccurlyeq$ is a partial order, and the set $IL(U, A, I)$ with the partial order $\preccurlyeq$ forms a complete lattice, called an interval-set concept lattice (for short, IS concept lattice) ([22]), in which, the supremum and infimum are defined as follows:

$$(\mathcal{X}_1, \mathcal{B}_1) \vee (\mathcal{X}_2, \mathcal{B}_2) = (gf(\mathcal{X}_1 \sqcup \mathcal{X}_2), \mathcal{B}_1 \sqcap \mathcal{B}_2),$$
$$(\mathcal{X}_1, \mathcal{B}_1) \wedge (\mathcal{X}_2, \mathcal{B}_2) = (\mathcal{X}_1 \sqcap \mathcal{X}_2, fg(\mathcal{B}_1 \sqcup \mathcal{B}_2)).$$

Obviously, $(\widehat{U}, \widehat{\emptyset_A})$, $(\widehat{\emptyset_U}, \widehat{A})$ are both the IS concepts for the formal context (U, A, I). □

3. Object-oriented interval-set concept lattices

By introducing the idea of lower and upper approximations in rough set theory into formal concept analysis, Yao [32,33] proposes the object-oriented and property-oriented concept lattices. Incomplete information makes the object-oriented concept uncertain or partially-known. Interval sets are used to describe the possible ranges of the extension and intension of the partially-known object-oriented concept. Since the lower and upper bounds of an interval set may not be the accurate extensions or intensions of some object-oriented concepts, it is difficult to obtain more precise object-oriented concept. In order to resolve it, a pair of approximate operators are defined between the two interval-set power sets, and object-oriented interval-set concept lattices are proposed to get the more accurate or certain object-oriented concepts.

3.1. Object-oriented concept lattices

For a formal context $(U,\ A,\ I)$, Duntsch [3,4] and Yao [32,33] define the necessity and possibility operators ${}^{\Box}, {}^{\Diamond}:\ 2^U \to 2^A$ as follows: for any $X \subseteq U$,

$$X^{\Box} = \{a \in A \mid a^{\downarrow} \subseteq X\},$$
$$X^{\Diamond} = \{a \in A \mid a^{\downarrow} \cap X \neq \emptyset\}. \tag{3.1}$$

Similarly, the necessity and possibility operators ${}^{\Box}, {}^{\Diamond}:\ 2^A \to 2^U$ are defined by: for any $B \subseteq A$,

$$B^{\Box} = \{x \in U \mid x^{\uparrow} \subseteq B\},$$
$$B^{\Diamond} = \{x \in U \mid x^{\uparrow} \cap B \neq \emptyset\}. \tag{3.2}$$

Let (U, A, I) be a formal context. For any $X,\ X_1,\ X_2 \subseteq U$ and $B,\ B_1,\ B_2 \subseteq A$, the following properties hold [3,4,32–34]:
(1) $X_1 \subseteq X_2 \Rightarrow X_1^{\Box} \subseteq X_2^{\Box},\ X_1^{\Diamond} \subseteq X_2^{\Diamond},\ B_1 \subseteq B_2 \Rightarrow B_1^{\Box} \subseteq B_2^{\Box},\ B_1^{\Diamond} \subseteq B_2^{\Diamond}$;
(2) $X^{\Box\Diamond} \subseteq X \subseteq X^{\Diamond\Box},\ B^{\Box\Diamond} \subseteq B \subseteq B^{\Diamond\Box}$;
(3) $X^{\Diamond\Box\Diamond} = X^{\Diamond},\ X^{\Box\Diamond\Box} = X^{\Box},\ B^{\Diamond\Box\Diamond} = B^{\Diamond},\ B^{\Box\Diamond\Box} = B^{\Box}$;
(4) $(X_1 \cap X_2)^{\Box} = X_1^{\Box} \cap X_2^{\Box},\ (X_1 \cup X_2)^{\Diamond} = X_1^{\Diamond} \cup X_2^{\Diamond}$,
$(B_1 \cap B_2)^{\Box} = B_1^{\Box} \cap B_2^{\Box},\ (B_1 \cup B_2)^{\Diamond} = B_1^{\Diamond} \cup B_2^{\Diamond}$;
(5) $B^{\Diamond} \subseteq X \Leftrightarrow B \subseteq X^{\Box},\ X \subseteq B^{\Box} \Leftrightarrow X^{\Diamond} \subseteq B$.

A pair $(X,\ B)$ with $X \subseteq U$ and $B \subseteq A$ is called an object-oriented formal concept (for short, an OO concept), if $X = B^{\Diamond}$ and $B = X^{\Box}$. X is called the OO extension and B is called the OO intension of the OO concept (X, B). Then for any $X \subseteq U$ and $B \subseteq A$, $(X^{\Box\Diamond}, X^{\Box})$ and $(B^{\Diamond}, B^{\Diamond\Box})$ are both OO concepts. Since (U, A, I) is regular, (U, A) and $(\emptyset_U, \emptyset_A)$ are OO concepts. The set of all OO concepts is denoted by

$$L_o(U,\ A,\ I) = \{(X,\ B) \mid X = B^{\Diamond},\ B = X^{\Box}\}.$$

For any OO concepts $(X_1,\ B_1), (X_2,\ B_2) \in L_o(U, A, I)$, a partial relation $\leq_o$ is defined by:

$$(X_1,\ B_1) \leq_o (X_2,\ B_2) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow B_1 \subseteq B_2.$$

Then $(L_o(U, A, I), \leq_o)$ is a partial order set, called an OO concept lattice [32,33], in which the meet $\wedge_o$ and join $\vee_o$ operations are defined as follows:

$$(X_1, B_1) \wedge_o (X_2, B_2) = ((X_1 \cap X_2)^{\Box\Diamond}, B_1 \cap B_2),$$
$$(X_1, B_1) \vee_o (X_2, B_2) = (X_1 \cup X_2, (B_1 \cup B_2)^{\Diamond\Box}). \tag{3.3}$$

Table 3.1
The formal concept (U, A, I) of Example 3.1.

U		a	b	c	d	e
1	Frog	1	1	0	0	0
2	Dog	0	1	0	0	0
3	Spike-weed	1	0	1	0	1
4	Reed	1	1	1	0	1
5	Bean	0	1	1	1	0

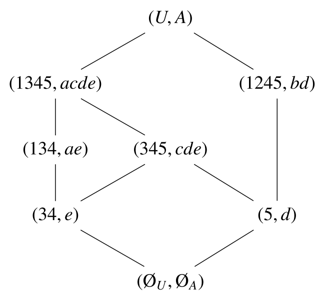

Fig. 2. Object-oriented concept lattice $L_o(U, A, I)$.

That is, for any OO concepts $(X_1, B_1), (X_2, B_2) \in L_o(U, A, I)$, there exists an OO concept smaller than (X_1, B_1) and (X_2, B_2), such that its intension is the meet of the two OO intensions of (X_1, B_1) and (X_2, B_2). On the other hand, there also exists a larger OO concept such that its extension is the join of the two OO extensions of (X_1, B_1) and (X_2, B_2). Then $L_o(U, A, I)$ is a complete lattice, called an OO concept lattice.

Similarly, for any subsets $X \subseteq U$ and $B \subseteq A$, the pair (X, B) is called a property-oriented formal concept (for short, an PO concept), if $X = B^{\Box}$ and $B = X^{\Diamond}$. X is called the PO extension and B is called the PO intension of the PO concept (X, B). Then for any $X \subseteq U$ and $B \subseteq A$, $(X^{\Diamond\Box}, X^{\Diamond})$ and $(B^{\Box}, B^{\Diamond\Box})$ are both PO concepts. The set of all PO concepts is denoted by

$$L_p(U, A, I) = \{(X, B) \mid X = B^{\Box}, B = X^{\Diamond}\}.$$

For any PO concepts $(X_1, B_1), (X_2, B_2) \in L_p(U, A, I)$, a partial relation $\leq_p$ is defined by:

$$(X_1, B_1) \leq_p (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow B_1 \subseteq B_2.$$

Then $(L_p(U, A, I), \leq_p)$ is a partial order set, called an PO concept lattice [32,33], in which the meet $\wedge_p$ and join $\vee_p$ operations are defined as follows:

$$\begin{aligned}(X_1, B_1) \wedge_p (X_2, B_2) &= (X_1 \cap X_2, (B_1 \cap B_2)^{\Box\Diamond}),\\ (X_1, B_1) \vee_p (X_2, B_2) &= ((X_1 \cup X_2)^{\Diamond\Box}, B_1 \cup B_2).\end{aligned} \tag{3.4}$$

For a formal context (U, A, I), applying Eqs. (3.1) and (3.2) we can get that [32,33]

$$X^{\Diamond} = [(X^c)^{\Box}]^c = X^{c\Box c},\ X^{\Box} = [(X^c)^{\Diamond}]^c = X^{c\Diamond c},\ B^{\Diamond} = [(B^c)^{\Box}]^c = B^{c\Box c},\ B^{\Box} = [(B^c)^{\Diamond}]^c = B^{c\Diamond c}, \tag{3.5}$$

where X^c denotes the complement of X. Furthermore, for the complement formal context (U, A, I^c) of (U, A, I) with $(x, a) \in I^c \Leftrightarrow (x, a) \notin I$, we denote by ${}^{\uparrow_c}$, ${}^{\downarrow_c}$, ${}^{\Box_c}$ and ${}^{\Diamond_c}$, respectively, the operators ${}^{\uparrow}$, ${}^{\downarrow}$, ${}^{\Box}$ and ${}^{\Diamond}$ in the complement formal context (U, A, I^c). Therefore,

$$(x, a) \in I^c \Leftrightarrow (x, a) \notin I \Leftrightarrow x \in a^{\downarrow_c} \Leftrightarrow a \in x^{\uparrow_c}.$$

According to Eqs. (2.8), (3.1) and (3.2) we also can obtain that [32,33]

$$X^{\Box} = (X^c)^{\uparrow_c},\ X^{\uparrow} = (X^c)^{\Box_c},\ B^{\Diamond} = (B^{\downarrow_c})^c,\ B^{\downarrow} = (B^{\Diamond_c})^c. \tag{3.6}$$

Example 3.1. Table 3.1 shows a formal context with $U = \{1, 2, 3, 4, 5\}$ and $A = \{a, b, c, d, e\}$, which is extracted from the context in the reference [30] to plan a Hungarian educational film entitled "Living Beings and Water". The objects in this table are the living beings mentioned in the film: 1: Frog, 2: Dog, 3:Spike-weed, 4: Reed, 5: Bean, and the attributes are the properties which the film emphasize: a: living in water, b: living on land, c: using chlorophyll to produce food, d: two seed leaves, e: one seed leaf.

The OO concept lattice of the formal context (U, A, I) given in Table 3.1 can be shown in Fig. 2.

3.2. Object-oriented interval-set concept lattices

For a partially-known OO concept, we give an interval set of objects to denote the possible range of the OO extension, and an interval set of attributes to depict the possible range of the OO intension. Since the OO extension and OO intension of any OO concept determine each other uniquely, a pair of operators is defined on the two interval sets to get the more accurate ranges of the OO extension and OO intension.

Definition 3.1. Let (U, A, I) be a formal context. For any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$, a pair $(\underline{f}, \overline{f})$ of operators between $\mathcal{I}(2^U)$ and $\mathcal{I}(2^A)$ is defined as follows:

$$\begin{aligned} \underline{f}(\mathcal{X}) &= [X_l^{\square},\ X_u^{\square}] = \{B \in 2^A \mid X_l^{\square} \subseteq B \subseteq X_u^{\square}\}, \\ \overline{f}(\mathcal{X}) &= [X_l^{\diamond},\ X_u^{\diamond}] = \{B \in 2^A \mid X_l^{\diamond} \subseteq B \subseteq X_u^{\diamond}\}. \end{aligned} \tag{3.7}$$

If $\mathcal{X}$ is the range of an extension of a partially-known OO (OP, respectively) concept, $\underline{f}(\mathcal{X})$ shows the range of the intension with $X_l^{\square}$ being the lower bound and $X_u^{\square}$ being the upper bound, and $\overline{f}(\mathcal{X})$ shows another range of the intension with $X_l^{\diamond}$ being the lower bound and $X_u^{\diamond}$ being the upper bound. And these two ranges of the intensions satisfy $\underline{f}(\mathcal{X}) \sqsubseteq \overline{f}(\mathcal{X})$.

And a pair $(\underline{g}, \overline{g})$ of operators between $\mathcal{I}(2^A)$ to $\mathcal{I}(2^U)$ is defined as follows:

$$\begin{aligned} \underline{g}(\mathcal{B}) &= [B_l^{\square},\ B_u^{\square}] = \{X \in 2^U \mid B_l^{\square} \subseteq X \subseteq B_u^{\square}\}, \\ \overline{g}(\mathcal{B}) &= [B_l^{\diamond},\ B_u^{\diamond}] = \{X \in 2^U \mid B_l^{\diamond} \subseteq X \subseteq B_u^{\diamond}\}. \end{aligned} \tag{3.8}$$

If $\mathcal{B}$ is the range of the intension of a partially-known OO (PO, respectively) concept, $\underline{g}(\mathcal{B})$ shows the range of the extension by using $B_l^{\square}$ as the lower bound and $B_u^{\square}$ as the upper bound, and $\overline{g}(\mathcal{B})$ shows another range of the extension by taking $B_l^{\diamond}$ as the lower bound and $B_u^{\diamond}$ as the upper bound. Furthermore, these two ranges of the extensions generated by the interval set $\mathcal{B}$ satisfy $\underline{g}(\mathcal{B}) \sqsubseteq \overline{g}(\mathcal{B})$.

Definition 3.2. Let (U, A, I) be a formal context. For any interval sets $\mathcal{X} \in \mathcal{I}(2^U)$ and $\mathcal{B} \in \mathcal{I}(2^A)$, if $\underline{f}(\mathcal{X}) = \mathcal{B}$ and $\overline{g}(\mathcal{B}) = \mathcal{X}$, we call the pair $(\mathcal{X}, \mathcal{B})$ an object-oriented interval-set formal concept (for short, an OOIS concept), where $\mathcal{X}$ is called the OOIS extension and $\mathcal{B}$ the OOIS intension of the OOIS concept $(\mathcal{X}, \mathcal{B})$. If $\overline{f}(\mathcal{X}) = \mathcal{B}$ and $\underline{g}(\mathcal{B}) = \mathcal{X}$, we call the pair $(\mathcal{X}, \mathcal{B})$ an property-oriented interval-set formal concept (for short, an POIS concept), where $\mathcal{X}$ is called the POIS extension and $\mathcal{B}$ the OOIS intension of the POIS concept $(\mathcal{X}, \mathcal{B})$.

Example 3.2. Let us consider the formal context (U, A, I) mentioned in Example 3.1. For a partially-known OOIS concept $C_o = (X, B)$, there exist many sets to denote the possible ranges of the OO extension X and the OO intension B, respectively.

(1) Take the interval set $\mathcal{X} = [3,\ 134]$ as the possible range of the OO extension X, and the interval set $\mathcal{B} = [e,\ cde]$ as the possible range of the OO intension B. The OO extension X must include the object 3, and may consist of the objects 1 or 4, and the OO intension B must include the attribute e, and may include the attributes c and e. That is,

$$\mathcal{X} = [3,\ 134] = \{3, 13, 34, 134\},\ \mathcal{B} = [e,\ cde] = \{e, ce, de, cde\}.$$

Any subset in $\mathcal{X}$ may be the OO extension, and any subset in $\mathcal{B}$ may be the OO intension. According to Definition 3.1 we can get that

$$\underline{f}(\mathcal{X}) = [3^{\square},\ 134^{\square}] = [\emptyset_A,\ ae] = \{\emptyset_A,\ a,\ e,\ ae\},\ \ \overline{g}(\mathcal{B}) = [e^{\diamond},\ cde^{\diamond}] = [34,\ 345] = \{34, 345\}.$$

$\underline{f}(\mathcal{X}) \neq \mathcal{B}$ makes the interval set $\mathcal{B}$ cannot include the OO intensions of some OO extensions in $\mathcal{X}$, and $\overline{g}(\mathcal{B}) \neq \mathcal{X}$ shows that some OO intensions in $\mathcal{B}$ cannot find the corresponding OO extensions in $\mathcal{X}$. That is to say, the ranges for the OO extension and the OO intension are not accurate.

(2) Choose another range $\mathcal{Y} = [34,\ 1345]$ for the OO extension X, and the range $\mathcal{C} = [e,\ acde]$ for the OO intension B. Then

$$\mathcal{Y} = [34,\ 1345] = \{34, 134, 345, 1345\},\ \ \mathcal{C} = [e,\ acde] = \{e, ae, ce, de, ace, ade, cde, acde\}.$$

Applying Definition 3.1 we can get that

$$\underline{f}(\mathcal{Y}) = [34^{\square},\ 1345^{\square}] = [e,\ acde] = \mathcal{C},\ \ \overline{g}(\mathcal{C}) = [e^{\diamond},\ acde^{\diamond}] = [34,\ 1345] = \mathcal{Y}.$$

Therefore, $\underline{f}(\mathcal{Y}) = \mathcal{C}$ and $\overline{g}(\mathcal{C}) = \mathcal{Y}$, and $(\mathcal{Y}, \mathcal{C})$ is an OOIS concept. Since $(34, e)$, $(134, ae)$ and $(1345, acde)$ are both OO concepts, and 34, 134, 1345 are the elements in the interval set $\mathcal{Y}$ and $e,\ ae$ and $acde$ belong to the interval set $\mathcal{C}$, we know that $(34, e)$, $(134, ae)$ or $(1345, acde)$ may be the OO concept $C_o = (X, B)$.

This example shows that, the real OO extension and OO intension of a partially-known OO concept exist in the OOIS extension and OOIS intension, respectively.

Property 3.1. Let (U, A, I) be a formal context. For any $\mathcal{X},\ \mathcal{X}_1,\ \mathcal{X}_2 \in \mathcal{I}(2^U)$ and $\mathcal{B},\ \mathcal{B}_1,\ \mathcal{B}_2 \in \mathcal{I}(2^A)$, the following properties hold:

(1) $\mathcal{X}_1 \sqsubseteq \mathcal{X}_2 \Rightarrow \underline{f}(\mathcal{X}_1) \sqsubseteq \underline{f}(\mathcal{X}_2),\ \overline{f}(\mathcal{X}_1) \sqsubseteq \overline{f}(\mathcal{X}_2)$,
$\mathcal{B}_1 \sqsubseteq \mathcal{B}_2 \Rightarrow \underline{g}(\mathcal{B}_1) \sqsubseteq \underline{g}(\mathcal{B}_2),\ \overline{g}(\mathcal{B}_1) \sqsubseteq \overline{g}(\mathcal{B}_2)$;
(2) $\overline{f}\underline{f}(\mathcal{X}) \sqsubseteq \mathcal{X} \sqsubseteq \underline{f}\overline{f}(\mathcal{X}),\ \overline{g}\underline{g}(\mathcal{B}) \sqsubseteq \mathcal{B} \sqsubseteq \underline{g}\overline{g}(\mathcal{B})$;
(3) $\overline{f}\underline{g}\overline{f}(\mathcal{X}) = \overline{f}(\mathcal{X}),\ \underline{f}\overline{g}\underline{f}(\mathcal{X}) = \underline{f}(\mathcal{X})$,
$\overline{g}\underline{f}\overline{g}(\mathcal{B}) = \overline{g}(\mathcal{B}),\ \underline{g}\overline{f}\underline{g}(\mathcal{B}) = \underline{g}(\mathcal{B})$;
(4) $\underline{f}(\mathcal{X}_1 \sqcap \mathcal{X}_2) = \underline{f}(\mathcal{X}_1) \sqcap \underline{f}(\mathcal{X}_2),\ \overline{f}(\mathcal{X}_1 \sqcup \mathcal{X}_2) = \overline{f}(\mathcal{X}_1) \sqcup \overline{f}(\mathcal{X}_2)$,
$\underline{g}(\mathcal{B}_1 \sqcap \mathcal{B}_2) = \underline{g}(\mathcal{B}_1) \sqcap \underline{g}(\mathcal{B}_2),\ \overline{g}(\mathcal{B}_1 \sqcup \mathcal{B}_2) = \overline{g}(\mathcal{B}_1) \sqcup \overline{g}(\mathcal{B}_2)$;
(5) $\overline{g}(\mathcal{B}) \sqsubseteq \mathcal{X} \Leftrightarrow \mathcal{B} \sqsubseteq \underline{f}(\mathcal{X}),\ \mathcal{X} \sqsubseteq \underline{g}(\mathcal{B}) \Leftrightarrow \overline{f}(\mathcal{X}) \sqsubseteq \mathcal{B}$.

Proof. According to the properties of the operators $^\Box$ and $^\Diamond$ on 2^U and 2^A, respectively, and Definition 3.1 we can directly obtain these results.

We denote by $IL_o(U, A, I)$ the set of all OOIS concepts of the formal context (U, A, I). A binary relation $\preceq_o$ on $IL_o(U, A, I)$ is defined as follows: for any $(\mathcal{X}_1, \mathcal{B}_1), (\mathcal{X}_2, \mathcal{B}_2) \in IL_o(U, A, I)$,

$$(\mathcal{X}_1, \mathcal{B}_1) \preceq_o (\mathcal{X}_2, \mathcal{B}_2) \Leftrightarrow \mathcal{X}_1 \sqsubseteq \mathcal{X}_2 \Leftrightarrow \mathcal{B}_1 \sqsubseteq \mathcal{B}_2. \tag{3.9}$$

$\preceq_o$ is a partial relation on $IL_o(U, A, I)$, and the partial set $(IL_o(U, A, I), \preceq_o)$ forms a complete lattice, called an object-oriented interval-set concept lattice (for short, OOIS concept lattice). The meet and join operations on it are defined by: for any OOIS concepts $(\mathcal{X}_1, \mathcal{B}_1), (\mathcal{X}_2, \mathcal{B}_2) \in IL_o(U, A, I)$,

$$\begin{aligned} (\mathcal{X}_1, \mathcal{B}_1) \wedge_o (\mathcal{X}_2, \mathcal{B}_2) &= (\overline{g}\underline{f}(\mathcal{X}_1 \sqcap \mathcal{X}_2), \mathcal{B}_1 \sqcap \mathcal{B}_2), \\ (\mathcal{X}_1, \mathcal{B}_1) \vee_o (\mathcal{X}_2, \mathcal{B}_2) &= (\mathcal{X}_1 \sqcup \mathcal{X}_2, \underline{f}\overline{g}(\mathcal{B}_1 \sqcup \mathcal{B}_2)). \end{aligned} \tag{3.10}$$

$IL_p(U, A, I)$ is the set of all POIS concepts for the formal context (U, A, I). For any $(\mathcal{X}_1, \mathcal{B}_1), (\mathcal{X}_2, \mathcal{B}_2) \in IL_p(U, A, I)$, a binary relation $\preceq_p$ on $IL_p(U, A, I)$ is defined by:

$$(\mathcal{X}_1, \mathcal{B}_1) \preceq_p (\mathcal{X}_2, \mathcal{B}_2) \Leftrightarrow \mathcal{X}_1 \sqsubseteq \mathcal{X}_2 \Leftrightarrow \mathcal{B}_1 \sqsubseteq \mathcal{B}_2. \tag{3.11}$$

Then $\preceq_p$ is a partial relation on $IL_p(U, A, I)$. The partial set $(IL_p(U, A, I), \preceq_p)$ forms a complete lattice, called a property-oriented interval-set concept lattice (for short, POIS concept lattice). And the meet and join operations on it are defined as follows: for any POIS concepts $(\mathcal{X}_1, \mathcal{B}_1), (\mathcal{X}_2, \mathcal{B}_2) \in IL_p(U, A, I)$,

$$\begin{aligned} (\mathcal{X}_1, \mathcal{B}_1) \wedge_p (\mathcal{X}_2, \mathcal{B}_2) &= (\mathcal{X}_1 \sqcap \mathcal{X}_2, \overline{f}\underline{g}(\mathcal{B}_1 \sqcap \mathcal{B}_2)), \\ (\mathcal{X}_1, \mathcal{B}_1) \vee_p (\mathcal{X}_2, \mathcal{B}_2) &= (\underline{g}\overline{f}(\mathcal{X}_1 \sqcup \mathcal{X}_2), \mathcal{B}_1 \sqcup \mathcal{B}_2). \end{aligned} \tag{3.12}$$

Applying Property 3.1 (3) we can get that, for any interval sets $\mathcal{X} \in \mathcal{I}(2^U)$ and $\mathcal{B} \in \mathcal{I}(2^A)$, $(\overline{g}\underline{f}(\mathcal{X})\underline{f}(\mathcal{X}))$ and $(\overline{g}(\mathcal{B}), \underline{f}\overline{g}(\mathcal{B}))$ are both OOIS concepts. Furthermore, since the formal context (U, A, I) is regular, one can get that $(U, A), (\emptyset_U, \emptyset_A) \in L_o(U, A, I)$. And for any $(X, B) \in L_o(U, A, I)$,

$$(\emptyset_U, \emptyset_A) \preceq_o (X, B) \preceq_o (U, A),$$

which implies that $(\widehat{U}, \widehat{A}),\ (\widehat{\emptyset_U}, \widehat{\emptyset_A}) \in IL_o(U, A, I)$, and for any OOIS concept $(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I)$,

$$(\widehat{\emptyset_U}, \widehat{\emptyset_A}) \preceq_o (\mathcal{X}, \mathcal{B}) \preceq_o (\widehat{U}, \widehat{A}).$$

Similarly, we can get that $(\widehat{U}, \widehat{A}),\ (\widehat{\emptyset_U}, \widehat{\emptyset_A}) \in IL_p(U, A, I)$, and for any POIS concept $(\mathcal{X}, \mathcal{B}) \in IL_p(U, A, I)$,

$$(\widehat{\emptyset_U}, \widehat{\emptyset_A}) \preceq_p (\mathcal{X}, \mathcal{B}) \preceq_p (\widehat{U}, \widehat{A}). \quad \Box$$

Theorem 3.1. *Let $(U,\ A,\ I)$ be a formal context. For any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$,*
(1) $(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I) \Leftrightarrow (X_l, B_l),\ (X_u, B_u) \in L_o(U, A, I)$ and $(X_l, B_l) \preceq_o (X_u, B_u)$;
(2) $(\mathcal{X}, \mathcal{B}) \in IL_p(U, A, I) \Leftrightarrow (X_l, B_l),\ (X_u, B_u) \in L_p(U, A, I)$ and $(X_l, B_l) \preceq_p (X_u, B_u)$.

Proof. Take any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$. If $(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I)$ is an OOIS concept, we can get that $\underline{f}(\mathcal{X}) = \mathcal{B}$ and $\overline{g}(\mathcal{B}) = \mathcal{X}$. Then

$$\begin{aligned}\underline{f}(\mathcal{X}) &= [X_l^{\Box}, X_u^{\Box}] = \mathcal{B} = [B_l, B_u],\\ \overline{g}(\mathcal{B}) &= [B_l^{\Diamond}, B_u^{\Diamond}] = \mathcal{X} = [X_l, X_u].\end{aligned}$$

Applying Eq. (2.4) for the equality of two interval sets, one has that $X_l^{\Box} = B_l$, $X_u^{\Box} = B_u$ and $B_l^{\Diamond} = X_l$, $B_u^{\Diamond} = X_u$. Thus, $(X_l, B_l), (X_u, B_u) \in L_o(U, A, I)$. Meanwhile, $[X_l, X_u] \in \mathcal{I}(2^U)$ implies that $X_l \subseteq X_u$, and then $(X_l, B_l) \le_o (X_u, B_u)$.

Suppose $(X_l, B_l), (X_u, B_u) \in L_o(U, A, I)$ and $(X_l, B_l) \le_o (X_u, B_u)$. Then $X_l \subseteq X_u$, $B_l \subseteq B_u$, and $[X_l, X_u] \in \mathcal{I}(2^U)$ and $[B_l, B_u] \in \mathcal{I}(2^A)$. According to $(X_l, B_l) \in L_o(U, A, I)$ we can get that $X_l^{\Box} = B_l$ and $B_l^{\Diamond} = X_l$. $(X_u, B_u) \in L_o(U, A, I)$ implies that $X_u^{\Box} = B_u$ and $B_u^{\Diamond} = X_u$. From which we can get that $\underline{f}(\mathcal{X}) = [X_l^{\Box}, X_u^{\Box}] = [B_l, B_u] = \mathcal{B}$, and $\overline{g}(\mathcal{B}) = [B_l^{\Diamond}, B_u^{\Diamond}] = [X_l, X_u] = \mathcal{X}$. By Definition 3.2 we get that $(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I)$.

Analogously, applying Eq. (2.4) and Definition 3.2 we can prove that (2) is true. □

Theorem 3.1 shows that any OOIS (POIS, respectively) concept constructs two OO (PO, respectively) concepts with a partial relation $\le_o$ ($\le_p$, respectively). Conversely, any two OO (PO, respectively) concepts with the partial order $\le_o$ ($\le_p$, respectively) can also produce an OOIS (POIS, respectively) concept.

For a formal context (U, A, I), we use the pair (f^c, g^c) to denote the operators f and g between $\mathcal{I}(2^U)$ and $\mathcal{I}(2^A)$ in the complement formal context (U, A, I^c). Then for any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$,

$$f^c(\mathcal{X}) = f^c([X_l, X_u]) = [X_u^{\uparrow_c}, X_l^{\uparrow_c}],\ g^c(\mathcal{B}) = g^c([B_l, B_u]) = [B_u^{\downarrow_c}, B_l^{\downarrow_c}]. \tag{3.13}$$

Thus, we get the following relationships among the IS concept lattices, OOIS concept lattices and POIS concept lattices:

Theorem 3.2. *Let (U, A, I) be a formal context. For any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$,*
(1) $(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I) \Leftrightarrow (\neg\mathcal{X}, \neg\mathcal{B}) \in IL_p(U, A, I)$;
(2) $(\mathcal{X}, \mathcal{B}) \in IL(U, A, I^c) \Leftrightarrow (\neg\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I) \Leftrightarrow (\mathcal{X}, \neg\mathcal{B}) \in IL_p(U, A, I)$.

Proof. (1) For any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$, we have $\neg\mathcal{X} = [X_u^c, X_l^c]$ and $\neg\mathcal{B} = [B_u^c, B_l^c]$. Applying Eqs. (3.5), (3.7) and (3.8) we can get that

$$\begin{aligned}\overline{f}(\neg\mathcal{X}) &= \overline{f}([X_u^c, X_l^c]) = [X_u^{c\Diamond}, X_l^{c\Diamond}] = [(X_u^{c\Diamond c})^c, (X_l^{c\Diamond c})^c]\\ &= [(X_u^{\Box})^c, (X_l^{\Box})^c] = \neg[X_l^{\Box}, X_u^{\Box}] = \neg\underline{f}([X_l, X_u]) = \neg\underline{f}(\mathcal{X}),\\ \underline{g}(\neg\mathcal{B}) &= \underline{g}([B_u^c, B_l^c]) = [B_u^{c\Box}, B_l^{c\Box}] = [(B_u^{c\Box c})^c, (B_l^{c\Box c})^c]\\ &= [(B_u^{\Diamond})^c, (B_l^{\Diamond})^c] = \neg[B_l^{\Diamond}, B_u^{\Diamond}] = \neg\overline{g}([B_l, B_u]) = \neg\overline{g}(\mathcal{B}).\end{aligned}$$

By Definition 3.2 we get that

$$\begin{aligned}(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I) &\Leftrightarrow \underline{f}(\mathcal{X}) = \mathcal{B},\ \overline{g}(\mathcal{B}) = \mathcal{X}\\ &\Leftrightarrow \overline{f}(\neg\mathcal{X}) = \neg\underline{f}(\mathcal{X}) = \neg\mathcal{B},\ \underline{g}(\neg\mathcal{B}) = \neg\overline{g}(\mathcal{B}) = \neg\mathcal{X}\\ &\Leftrightarrow (\neg\mathcal{X}, \neg\mathcal{B}) \in IL_p(U, A, I).\end{aligned}$$

(2) According to Eqs. (3.6), (3.7), (3.8), and (3.13) we have

$$\begin{aligned}\underline{f}(\mathcal{X}) &= [X_l^{\Box}, X_u^{\Box}] = [(X_l^c)^{\uparrow_c}, (X_u^c)^{\uparrow_c}] = f^c([X_u^c, X_l^c]) = f^c(\neg[X_l, X_u]) = f^c(\neg\mathcal{X}),\\ \overline{g}(\mathcal{B}) &= [B_l^{\Diamond}, B_u^{\Diamond}] = [(B_l^{\downarrow_c})^c, (B_u^{\downarrow_c})^c] = \neg[B_u^{\downarrow_c}, B_l^{\downarrow_c}] = \neg g^c([B_l, B_u]) = \neg g^c(\mathcal{B}).\end{aligned}$$

Thus,

$$\begin{aligned}(\mathcal{X}, \mathcal{B}) \in IL(U, A, I^c) &\Leftrightarrow f^c(\mathcal{X}) = \mathcal{B},\ g^c(\mathcal{B}) = \mathcal{X}\\ &\Leftrightarrow \underline{f}(\neg\mathcal{X}) = f^c(\mathcal{X}) = \mathcal{B},\ \overline{g}(\mathcal{B}) = \neg g^c(\mathcal{B}) = \neg\mathcal{X}\\ &\Leftrightarrow (\neg\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I)\\ &\Leftrightarrow (\mathcal{X}, \neg\mathcal{B}) \in IL_p(U, A, I). \quad \Box\end{aligned}$$

Theorem 3.2 shows that the OOIS concept lattice $IL_o(U, A, I)$ and POIS concept lattice $IL_p(U, A, I)$ are isomorphic. And the IS concept lattice $IL(U, A, I^c)$ of the complement formal context (U, A, I^c) is isomorphic to the OOIS (POIS, respectively) concept lattice $IL_o(U, A, I)$ ($IL_p(U, A, I)$, respectively) of the formal context (U, A, I).

4. Approaches to construct OOIS concept lattices

An interval set of objects (attributes, respectively) is used to describe the possible range of the OO extension (OO intension, respectively) for a partially-known OO concept. Then an OOIS concept describes the more accurate ranges of the OO extension and OO intension for the partially-known OO concept. In this section, we investigate the relationships between OO concept lattices and OOIS concept lattices, and provide the approaches to generate an OOIS concept lattice.

Theorem 4.1. *Let (U, A, I) be a formal context. $L_o(U, A, I)$ is the OO concept lattice, and $IL_o(U, A, I)$ is the OOIS concept lattice. Then*

$$IL_o(U, A, I) = \{([X, Y], [B, C]) \mid (X, B), (Y, C) \in L_o(U, A, I),\ X \subseteq Y\}.$$

Proof. It should be noted that $L = \{([X, Y], [B, C]) \mid (X, B), (Y, C) \in L_o(U, A, I),\ X \subseteq Y\}$. For any $(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I)$ with $\mathcal{X} = [X_l, X_u]$ and $\mathcal{B} = [B_l, B_u]$, applying Theorem 3.1 we can get that $(X_l, B_l),\ (X_u, B_u) \in L_o(U, A, I)$ and $(X_l, B_l) \preceq_o (X_u, B_u)$. Thus, $X_l \subseteq X_u$ and $(\mathcal{X}, \mathcal{B}) = ([X_l, X_u], [B_l, B_u]) \in L$. By the arbitrariness of $(\mathcal{X}, \mathcal{B})$ we can get that $IL_o(U, A, I) \subseteq L$.

Take any $([X, Y], [B, C]) \in L$. We have $(X, B), (Y, C) \in L_o(U, A, I)$ and $X \subseteq Y$. Thus, $(X, B) \preceq_o (Y, C)$. Again by using Theorem 3.1 we can get that $([X, Y], [B, C]) \in IL_o(U, A, I)$. The arbitrariness of $([X, Y], [B, C])$ implies that $L \subseteq IL_o(U, A, I)$. All these induce that $IL_o(U, A, I) = L$. □

Theorem 4.2. *Let (U, A, I) be a formal context. For any OO concept $(X, B) \in L_o(U, A, I)$, we have*

(1) $(\widehat{X}, \widehat{B}) \in IL_o(U, A, I)$;

(2) $([\emptyset_U, X], [\emptyset_A, B]) \in IL_o(U, A, I)$, $([X, U], [B, A]) \in IL_o(U, A, I)$, and

$$(\widehat{\emptyset}_U, \widehat{\emptyset}_A) \preceq_o ([\emptyset_U, X], [\emptyset_A, B]) \preceq_o (\widehat{X}, \widehat{B}) \preceq_o ([X, U], [B, A])) \preceq_o (\widehat{U}, \widehat{A}).$$

Proof. For any OO concepts $(X, B) \in L_o(U, A, I)$, we have $X^\square = B$ and $B^\diamond = X$.

(1) Since $(\widehat{X}, \widehat{B}) = ([X, X], [B, B])$ and $(X, B) \in L_o(U, A, I)$, one has

$$\underline{f}(\widehat{X}) = \underline{f}([X, X]) = [X^\square, X^\square] = [B, B] = \widehat{B},$$
$$\overline{g}(\widehat{B}) = \overline{g}([B, B]) = [B^\diamond, B^\diamond] = [X, X] = \widehat{X}.$$

Therefore, $(\widehat{X}, \widehat{B}) \in IL_o(U, A, I)$.

(2) Since for the formal context (U, A, I), $(\emptyset_U, \emptyset_A), (U, A) \in L_o(U, A, I)$, one has $\emptyset_U^\square = \emptyset_A$, $\emptyset_A^\diamond = \emptyset_U$, $U^\square = A$ and $A^\diamond = U$. Then for any $(X, B) \in L_o(U, A, I)$,

$$\underline{f}([\emptyset_U, X]) = [\emptyset_U^\square, X^\square] = [\emptyset_A, B], \quad \overline{g}([\emptyset_A, B]) = [\emptyset_A^\diamond, B^\diamond] = [\emptyset_U, X],$$
$$\underline{f}([X, U]) = [X^\square, U^\square] = [B, A], \quad \overline{g}([B, A]) = [B^\diamond, A^\diamond] = [X, U].$$

That is to say, $([\emptyset_U, X], [\emptyset_A, B])$ and $([X, U], [B, A])$ are both OOIS concepts. Furthermore, $\emptyset_U \subseteq X \subseteq U$, $\emptyset_A \subseteq B \subseteq A$ and Eq. (3.9) imply that

$$(\widehat{\emptyset}_U, \widehat{\emptyset}_A) \preceq_o ([\emptyset_U, X], [\emptyset_U, B]) \preceq_o (\widehat{X}, \widehat{B}) \preceq_o ([X, U], [B, A])) \preceq_o (\widehat{U}, \widehat{A}). \quad \square$$

Theorem 4.3. *Let (U, A, I) be a formal context, and $IL_o(U, A, I)$ the OOIS concept lattice of (U, A, I). It should be noted that:*

$$IL_o^l(U, A, I) = \{([\emptyset_U, X], [\emptyset_A, B]) \mid (X, B) \in L_o(U, A, I)\},$$
$$IL_o^s(U, A, I) = \{(\widehat{X}, \widehat{B}) \mid (X, B) \in L_o(U, A, I)\},$$
$$IL_o^u(U, A, I) = \{([X, U], [B, A]) \mid (X, B) \in L_o(U, A, I)\},$$
$$IL_o^p(U, A, I) = \{([X, Y], [B, C]) \mid (X, B), (Y, C) \in L_o(U, A, I),\ X \subseteq Y,\ X \neq \emptyset_U,\ Y \neq U,\ X \neq Y\}.$$

Then

$$IL_o(U, A, I) = IL_o^l(U, A, I) \cup IL_o^s(U, A, I) \cup IL_o^u(U, A, I) \cup IL_o^p(U, A, I).$$

Proof. Take any OOIS concept $(\mathcal{X}, \mathcal{B}) \in IL_o(U, A, I)$ with $\mathcal{X} = [X_l, X_u]$ and $\mathcal{B} = [B_l, B_u]$. By Theorem 3.1 we have $(X_l, B_l),\ (X_u, B_u) \in L_o(U, A, I)$, $\emptyset_U \subseteq X_l \subseteq X_u \subseteq U$ and $\emptyset_A \subseteq B_l \subseteq B_u \subseteq A$.

(1) If $X_l = \emptyset_U$, $(X_l, B_l) \in L_o(U, A, I)$ implies that $B_l = \emptyset_A$. And $(\mathcal{X}, \mathcal{B}) = ([X_l, X_u], [B_l, B_u]) = ([\emptyset_U, X_u], [\emptyset_A, B_u]) \in IL_o^l(U, A, I)$.

(2) If $X_l = U$, $(X_l, B_l) \in L_o(U, A, I)$ implies that $B_l = A$. And $(\mathcal{X}, \mathcal{B}) = ([X_l, X_u], [B_l, B_u]) = ([X_l, U], [B_l, A]) \in IL_o^u(U, A, I)$.

(3) If $X_l = X_u$, $(X_l, B_l),\ (X_u, B_u) \in L_o(U, A, I)$ imply that $(X_l, B_l) = (X_u, B_u)$ and $(\mathcal{X}, \mathcal{B}) = ([X_l, X_l], [B_l, B_l]) = (\widehat{X_l}, \widehat{B_l}) \in IL_o^s(U, A, I)$.

(4) If $X_l \neq \emptyset_U$, $X_l \neq U$ and $X_l \neq X_u$, we have $(X_l, B_l) \neq (X_u, B_u)$. Together with $(X_l, B_l),\ (X_u, B_u) \in L_o(U, A, I)$ and $X_l \subseteq X_u$ we can obtain that $(\mathcal{X}, \mathcal{B}) = ([X_l, X_u], [B_l, B_u]) \in IL_o^p(U, A, I)$.

All these shows that $IL_o(U, A, I) \subseteq IL_o^l(U, A, I) \cup IL_o^s(U, A, I) \cup IL_o^u(U, A, I) \cup IL_o^p(U, A, I)$. Based on Theorem 4.1 and Theorem 4.2 we also obtain $IL_o^l(U, A, I) \cup IL_o^s(U, A, I) \cup IL_o^u(U, A, I) \cup IL_o^p(U, A, I) \subseteq IL_o(U, A, I)$. Therefore, $IL_o(U, A, I) = IL_o^l(U, A, I) \cup IL_o^s(U, A, I) \cup IL_o^u(U, A, I) \cup IL_o^p(U, A, I)$. □

Remark 4.1. Theorem 4.3 divides the OOIS concept lattice $IL_o(U,A,I)$ into four parts $IL_o^l(U,A,I)$, $IL_o^s(U,A,I)$, $IL_o^u(U,A,I)$ and $IL_o^p(U,A,I)$.

(1) $IL_o^l(U,A,I)$ is the set of OOIS concepts generated by the OO concept $(\emptyset_U,\emptyset_A)$ and any OO concept (X,B) with $X\subseteq U$ and $B\subseteq A$. The lower bounds of the OOIS extension and OOIS intension for any OOIS concept in $IL_o^l(U,A,I)$ are just the OO extension and OO intension, respectively, of the OO concept $(\emptyset_U,\emptyset_A)$.

(2) $IL_o^s(U,A,I)$ is the set of OOIS concepts constructed by all OO concepts. The pair of the single interval sets generated by the OO extension and OO intension for any OO concept in $L_o(U,A,I)$ is an OOIS concept in $IL_o^s(U,A,I)$.

(3) $IL_o^u(U,A,I)$ is the set of OOIS concepts constructed by any OO concept (X,B) and the OO concept (U,A) with $X\subseteq U$ and $B\subseteq A$. The two upper bounds of the OOIS extension and OOIS intension for any OOIS concept are the OO extension and intension, respectively, of the OO concept (U,A).

(4) $IL_o^p(U,A,I)$ is the set of OOIS concepts constructed by any two OO concepts (X,B) and (Y,C) with $X\subseteq Y$, $X\neq\emptyset_U$, $Y\neq U$ and $X\neq Y$.

According to the definitions of $IL_o^l(U,A,I)$, $IL_o^s(U,A,I)$, $IL_o^u(U,A,I)$ and $IL_o^p(U,A,I)$ given in Theorem 4.3, we can get that

$$\begin{aligned} IL_o^l(U,A,I)\cap IL_o^s(U,A,I) &= \{\,(\widehat{\emptyset}_U,\widehat{\emptyset}_A)\,\},\\ IL_o^u(U,A,I)\cap IL_o^s(U,A,I) &= \{\,(\widehat{U},\widehat{A})\,\},\\ IL_o^l(U,A,I)\cap IL_o^u(U,A,I) &= \{\,([\emptyset_U,U][\emptyset_A,A])\,\}. \end{aligned} \tag{4.1}$$

Theorem 4.4. *Let (U,A,I) be a formal context, $L_o(U,A,I)$ the OO concept lattice, and $IL_o(U,A,I,)$ the OOIS concept lattice of (U,A,I).*

(1) $IL_o^l(U,A,I)$, $IL_o^s(U,A,I)$ and $IL_o^u(U,A,I)$ are all sublattices of $IL_o(U,A,I)$ with the meet $\wedge_o$ and join $\vee_o$;
(2) $IL_o^l(U,A,I)\cong L_o(U,A,I)$;
(3) $IL_o^s(U,A,I)\cong L_o(U,A,I)$;
(4) $IL_o^u(U,A,I)\cong L_o(U,A,I)$;

Proof. (1) The partial set $(IL_o(U,A,I),\preceq_o)$ is a complete lattice with the meet $\wedge_o$ and join $\vee_o$ defined in Eq. (3.10). For any OOIS concepts $([\emptyset_U,X],[\emptyset_A,B])$, $([\emptyset_U,Y],[\emptyset_A,C])\in IL_o^l(U,A,I)$ with (X,B), $(Y,C)\in L_o(U,A,I)$, according to Eq. (3.10) we get that

$$\begin{aligned} ([\emptyset_U,X],[\emptyset_A,B])\wedge_o([\emptyset_U,Y],[\emptyset_A,C]) &= (\,\overline{\underline{g}f}(\,[\emptyset_U,X]\sqcap[\emptyset_U,Y]\,),[\emptyset_A,B]\sqcap[\emptyset_A,C]\,)\\ &= (\,\overline{\underline{g}f}(\,[\emptyset_U,X\cap Y]\,),[\emptyset_A,B\cap C]\,)\\ &= (\,[\emptyset_U^{\Box\Diamond},\ (X\cap Y)^{\Box\Diamond}],\ [\emptyset_A,B\cap C]\,). \end{aligned}$$

Applying Eq. (3.4), (X,B), $(Y,C)\in L_o(U,A,I)$ imply that $(X,B)\wedge_o(Y,C)=((X\cap Y)^{\Box\Diamond},B\cap C)\in L_o(U,A,I)$. And $(\emptyset_U,\emptyset_A)\in L_o(U,A,I)$ implies $\emptyset_U^{\Box\Diamond}=\emptyset_U$. Thus,

$$\begin{aligned} ([\emptyset_U,X],[\emptyset_A,B])\wedge_o([\emptyset_U,Y],[\emptyset_A,C]) &= (\,[\emptyset_U^{\Box\Diamond},\ (X\cap Y)^{\Box\Diamond}],\ [\emptyset_A,B\cap C]\,)\\ &= (\,[\emptyset_U,\ (X\cap Y)^{\Box\Diamond}],\ [\emptyset_A,B\cap C]\,)\in IL_o^l(U,A,I). \end{aligned}$$

Similarly, we can prove

$$\begin{aligned} ([\emptyset_U,X],[\emptyset_A,B])\vee_o([\emptyset_U,Y],[\emptyset_A,C]) &= (\,[\emptyset_U,\ X\cup Y],\ [\emptyset_A^{\Diamond\Box},(B\cup C)^{\Diamond\Box}]\,)\\ &= (\,[\emptyset_U,\ X\cup Y],\ [\emptyset_A,(B\cup C)^{\Diamond\Box}]\,)\in IL_o^l(U,A,I). \end{aligned}$$

Therefore, $IL_o^l(U,A,I)$ is a sublattice of $IL_o(U,A,I)$ with the meet $\wedge_o$ and the join $\vee_o$ in Eq. (3.10).

Analogously, we can prove that $IL_o^s(U,A,I)$ and $IL_o^u(U,A,I)$ are the sublattices of $IL_o(U,A,I)$ with the meet $\wedge_o$ and the join $\vee_o$ in Eq. (3.10).

(2) A mapping $\varphi:\ L_o(U,A,I)\to IL_o^l(U,A,I)$ is defined as follows: for any $(X,B)\in L_o(U,A,I)$,

$$\varphi((X,B))=(\,[\emptyset_U,X],\ [\emptyset_A,B]).$$

It is obvious that φ is a bijection mapping. By the proof in (1), Eqs. (3.4) and (3.10) we can get that, for any (X,B), $(Y,C)\in L_o(U,A,I)$,

$$\begin{aligned} \varphi((X,B)\wedge_o(Y,C)) &= \varphi((X\cap Y)^{\Box\Diamond},B\cap C)=(\,[\emptyset_U,(X\cap Y)^{\Box\Diamond}],[\emptyset_A,B\cap C]\,),\\ \varphi((X,B))\wedge_o\varphi((Y,C)) &= ([\emptyset_U,X],[\emptyset_A,B])\wedge_o([\emptyset_U,Y],[\emptyset_A,C])=(\,[\emptyset_U,(X\cap Y)^{\Box\Diamond}],[\emptyset_A,B\cap C]\,). \end{aligned}$$

Then

$$\varphi((X,B)\wedge_o(Y,C))=\varphi((X,B))\wedge_o\varphi((Y,C)).$$

Thus, the mapping φ is isomorphic between $L_o(U,A,I)$ and $IL_o^l(U,A,I)$, i.e. $L_o(U,A,I)\cong IL_o^l(U,A,I)$.

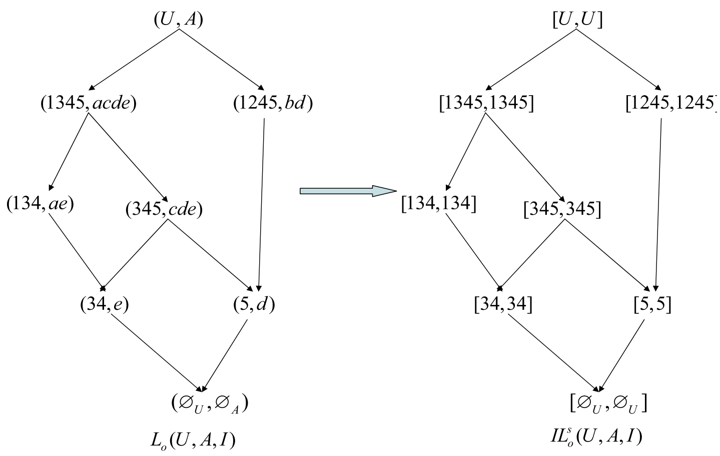

Fig. 3. $L_o(U, A, I)$ and $IL_o^s(U, A, I)$.

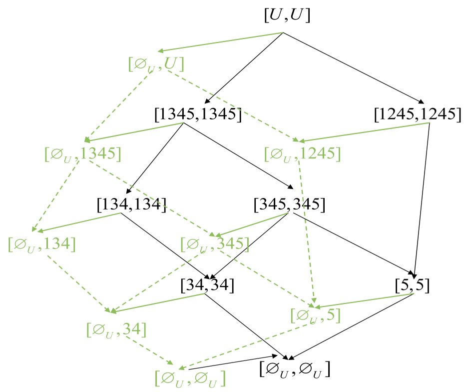

Fig. 4. $IL_o^l(U, A, I)$ and $IL_o^s(U, A, I)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Analogously, we can prove (3) and (4) are true.

According to Theorem 3.1, Theorems 4.1, 4.2, 4.3 and 4.4, we can show the approach to construct the OOIS concept lattice $IL_o(U, A, I)$ according to the OO concept lattice $L_o(U, A, I)$:

Step 1. Construct $IL_o^s(U, A, I) = \{(\widehat{X}, \widehat{B}) \mid (X, B) \in L_o(U, A, I)\}$ by using the single interval sets $\widehat{X}$ and $\widehat{B}$ instead of the OO concept (X, B).

Theorem 4.4 (3) tells us that $IL_o^s(U, A, I)$ is isomorphic to $L_o(U, A, I)$. Changing any OO extension in $L_o(U, A, I)$ to a singe interval set, we obtain all corresponding OOIS extensions. For the formal context (U, A, I) given in Table 3.1, Fig. 3 shows $L_o(U, A, I)$ and $IL_o^s(U, A, I)$. For simplicity, we just show the OOIS extensions in the figure of $IL_o^s(U, A, I)$.

Step 2. Construct $IL_o^l(U, A, I) = \{([\emptyset_U, X], [\emptyset_A, B]) \mid (X, B) \in L_o(U, A, I)\}$ by using the interval sets $[\emptyset_U, X]$ and $[\emptyset_A, B]$ as the OOIS extension and OOIS intension, respectively.

Theorem 4.4 (2) shows that $IL_o^l(U, A, I) \cong L_o(U, A, I)$. Generating the interval sets $[\emptyset_U, X]$ and $[\emptyset_A, B]$ for any $(X, B) \in L_o(U, A, I)$, we can get all OOIS concepts $([\emptyset_U, X], [\emptyset_A, B])$ in $IL_o^l(U, A, I)$. Theorem 4.4 (2) and (3) show that $IL_o^l(U, A, I) \cong IL_o^s(U, A, I)$. Fig. 4 shows $IL_o^l(U, A, I)$ with the green line and $IL_o^s(U, A, I)$ with the black line.

Step 3. Construct $IL_o^u(U, A, I) = \{ \ (\ [X, U], [B, A] \ ) \mid (X, B) \in L_o(U, A, I) \ \}$ with the interval sets $[X, U]$ and $[B, A]$ being the OOIS extension and OOIS intension, respectively.

Theorem 4.4 (4) shows that $IL_o^u(U, A, I) \cong L_o(U, A, I)$. Producing the interval sets $[X, U]$ and $[B, A]$ for any $(X, B) \in L_o(U, A, I)$, we can get all OOIS concepts $([X, U], [B, A])$ in $IL_o^u(U, A, I)$. Theorem 4.4 (3) and (4) show that $IL_o^u(U, A, I) \cong IL_o^s(U, A, I)$. Fig. 5 displays $IL_o^l(U, A, I)$ with the green line, $IL_o^s(U, A, I)$ with the black line and $IL_o^u(U, A, I)$ with the blue line.

Step 4. Construct $IL_o^p(U, A, I) = \{([X, Y], [B, C]) \mid (X, B), (Y, C) \in L_o(U, A, I), \ X \subseteq Y, X \neq \emptyset_U, \ Y \neq U, \ X \neq Y\}$ by choosing the interval sets $[X, Y]$ and $[B, C]$ being the OOIS extension and OOIS intension, respectively.

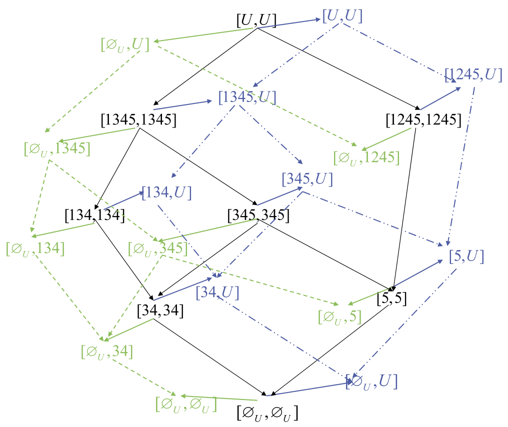

Fig. 5. $IL_o^l(U,A,I)$, $IL_o^s(U,A,I)$ and $IL_o^u(U,A,I)$.

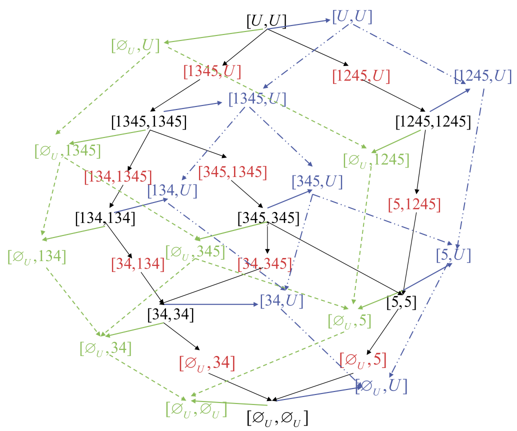

Fig. 6. $IL_o^l(U,A,I)$, $IL_o^s(U,A,I)$, $IL_o^u(U,A,I)$ and $IL_o^p(U,A,I)$.

$X \neq \emptyset_U$, $Y \neq U$ and $X \neq Y$ mean that $(\widehat{X},\widehat{B}) \preceq_o ([X,Y],[B,C]) \preceq_o (\widehat{Y},\widehat{C})$. Then any OOIS $([X,Y],[B,C])$ in $IL_o^p(U,A,I)$ appears between two different OOIS concepts $(\widehat{X},\widehat{B})$ and $(\widehat{Y},\widehat{C})$ in $IL_o^s(U,A,I)$. Fig. 6 shows elements in $IL_o^p(U,A,I)$ with the red color.

Step 5. Deleting the OOIS concepts $([U,U],[A,A])$ in $IL_o^s(U,A,I) \cap IL_o^u(U,A,I)$, $([\emptyset_U,\emptyset_U],[\emptyset_A,\emptyset_A])$ in $IL_o^s(U,A,I) \cap IL_o^l(U,A,I)$, and $([\emptyset_U,U],[\emptyset_A,A])$ in $IL_o^l(U,A,I) \cap IL_o^u(U,A,I)$, and reordering all OOIS concepts under the partial order $\preceq_o$, we can get the OOIS concept lattice in Fig. 7.

For a formal context (U,A,I), $L_o(U,A,I)$ is the OO concept lattice. And for any $X \subseteq U$ and $B \subseteq A$, $(X,B) \in L_o(U,A,I)$ is an OO concept if and only if $X^\Box = B$ and $B^\Diamond = X$. Now, in order to obtain the OOIS concepts lattice $IL_o(U,A,I)$, we take any interval sets $\mathcal{X} = [X_l, X_u] \in \mathcal{I}(2^U)$ and $\mathcal{B} = [B_l, B_u] \in \mathcal{I}(2^A)$ to verify $\underline{f}(\mathcal{X}) = \mathcal{B}$ and $\overline{g}(\mathcal{B}) = \mathcal{X}$. That is, $X_l^\Box = B_l$, $B_l^\Diamond = X_l$, and $X_u^\Box = B_u$, $B_u^\Diamond = X_u$. On the other hand, for the formal context (U,A,I), there exist $2^{2|U|}$ interval sets of objects in $\mathcal{I}(2^U)$, and $2^{2|A|}$ interval sets of attributes in $\mathcal{I}(2^A)$. Then the complexity of the algorithm to vary the pair of interval sets of objects and attributes such as $(\mathcal{X},\mathcal{B})$ is $O(2^{2|U|} \times 2^{2|A|})$. It is necessity to reduce the process. Theorems 4.1, 4.3 and 4.4 show that we just need to compute $IL_o^p(U,A,I)$, and the other three parts $IL_o^l(U,A,I)$, $IL_o^s(U,A,I)$, and $IL_o^u(U,A,I)$ are all isomorphic to $IL_o(U,A,I)$. Then the OOIS concept lattice $IL_o(U,A,I)$ is easily constructed just by the OO concept lattice $L_o(U,A,I)$. $\Box$

5. Conclusion

It has been argued in this paper that obtaining and constructing object-oriented interval-set concept lattice is essential to the knowledge discovery and attribute reduction for a formal context. Based on the theories of object-oriented concept lattices and interval sets, a new concept lattice, called object-oriented interval-set concept lattice, has been introduced to describe the partially-known OO concepts more accurate. Related properties of them are discussed. According to the structures of OO concepts and OOIS concepts, the relationships between OO concept lattices and OOIS concept lattices are

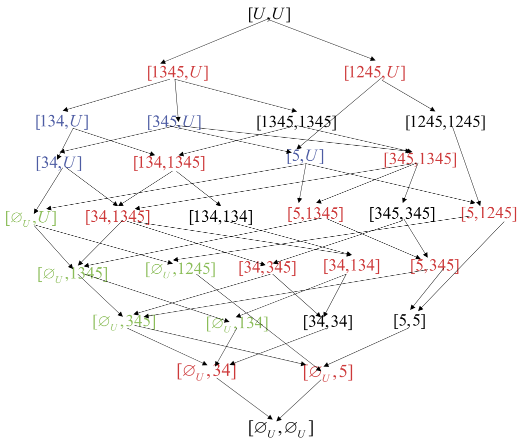

Fig. 7. $IL_o(U, A, I)$.

proposed. By dividing the OOIS concept lattice into four different parts, three of them are isomorphic to the OO concept lattice. Then the algorithm to construct the OOIS concept lattice according to the OO concept lattice is discussed.

In the future, we will discuss the interval-set concept lattice for a decision formal context, and investigate the related rule acquisition.

Conflict of interest statement

There is no conflict of interest statement.

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