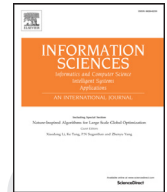




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## Three-way cognitive concept learning via multi-granularity

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### ABSTRACT

The key strategy of the three-way decisions theory is to consider a decision-making problem as a ternary classification one (i.e. acceptance, rejection and non-commitment). Recently, this theory has been introduced into formal concept analysis for mining three-way concepts to support three-way decisions in formal contexts. That is, the three-way decisions have been performed by incorporating the idea of ternary classification into the design of extension or intension of a concept. However, the existing methods on the studies of three-way concepts are constructive, which means that the three-way concepts had been formed by defining certain concept-forming operators in advance. In order to reveal the essential characteristics of three-way concepts in making decisions from the perspective of cognition, it is necessary to reconsider three-way concepts under the framework of general concept-forming operators. In other words, axiomatic approaches are required to characterize three-way concepts. Motivated by this problem, this study mainly focuses on three-way concept learning via multi-granularity from the viewpoint of cognition. Specifically, we firstly put forward an axiomatic approach to describe three-way concepts by means of multi-granularity. Then, we design a three-way cognitive computing system to find composite three-way cognitive concepts. Furthermore, we use the idea of set approximation to simulate cognitive processes for learning three-way cognitive concepts from a given clue. Finally, numerical experiments are conducted to evaluate the performance of the proposed learning methods.

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### 1. Introduction

Three-way decisions are one of the important ways in solving decision-making problems. Their key strategy is to consider a decision-making problem as a ternary classification one labeled by acceptance, rejection and non-commitment [60]. Up to now, substantial contributions have been made to the development of the theory of three-way decisions from various aspects. For instance, Yao [58] discussed the induction of three-way decision rules using the classical and decision-theoretic rough set models, and he also expounded the superiority of three-way decisions from the perspective of miss-classification cost [59]. Yang and Yao [53] employed the decision-theoretic rough set to model multi-agent three-way decisions. Deng and Yao [6] proposed a three-way approximation of a fuzzy set by means of the two

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9 parameters in the fuzzy membership function. Hu [9] established axiomatic approaches for three-way decisions and their  
10 corresponding spaces. Liang and Liu [19] built three-way decisions for the purpose of solving single-period and multi-  
11 period decision-making problems under intuitionistic fuzzy environment. Liu et al. [51] derived three-way decisions from  
12 investment decision-making problem for maximizing profit. In addition, three-way decisions have been applied to spam  
13 e-mail filtering [12], cost-sensitive face recognition [18], recommender system design [65], clustering analysis [67], and  
14 so on [68].

15 Cognitive computing is known as a computer system modeled on the human brain [42]. Its main purpose is to simulate  
16 human thought processes (e.g. perception, attention and learning) by computers. Cognitive learning is the function used to  
17 simulate cognitive processes such as the operations of thinking and remembering something. Generally speaking, cognitive  
18 learning can be viewed as a mathematical tool for the realization of cognitive computing. Moreover, both cognitive comput-  
19 ing and cognitive learning have absorbed many novel methods from psychology, information theory and mathematics in the  
20 process of their development [41].

21 A concept, generally constituted by its extension and intension parts, is the basic unit of human cognition in philos-  
22 ophy [41], and is commonly used to recognize a real-world concrete entity or model a perceived-world abstract subject  
23 [42]. Up to now, many types of concepts such as abstract concepts [41], Wille's concepts [46], property-oriented concepts  
24 [7], object-oriented concepts [55,56], AFS-concepts [43] and approximate concepts [15] have been presented to meet dif-  
25 ferent requirements of cognitive knowledge discovery. These well defined concepts can be distinguished from one another  
26 according to the characteristics of their intensions whose forms may be conjunctive, disjunctive or mixed. Very recently,  
27 by combining the theory of three-way decisions with formal concept analysis, Qi et al. [31,32] proposed the notion of a  
28 three-way concept to support three-way decisions in formal contexts, in which the main strategy is to incorporate the idea  
29 of ternary classification into the design of extension or intension of a concept. However, the existing methods on the studies  
30 of three-way concepts are constructive, which means that the three-way concepts were generated by introducing certain  
31 concept-forming operators in advance. In other words, researchers may define different three-way concepts with different  
32 properties, which results in a problem that which properties are the intrinsic ones of characterizing three-way concepts. The  
33 answers on this problem are important because they can help to understand the most basic decision-making mechanism of  
34 three-way concepts. So, axiomatic methods are required to look beyond appearance for the essence of three-way concepts  
35 in making decisions. The main theme of our paper is to address this problem.

36 Concept learning is to adopt certain approaches to learn unknown concepts from a given clue such as concept algebra  
37 system [41], queries [1], cognitive system [66], cloud model [44], set approximation [16], iteration [35], etc. According to  
38 Yao's information processing triangle [57], concept learning can be investigated from three aspects: the abstract level, brain  
39 level and machine level. More specifically, concept learning in the abstract level is to be analyzed in philosophy, mathematics  
40 and logics. For example, the formalization of the notion of a concept often refers to the principles from philosophy [14], the  
41 establishment of general concept-forming operators needs axiomatic methods [23], and logics are beneficial to the design  
42 of coherent cognitive systems. Concept learning in the brain level is to be discussed in psychology and neuroscience. For  
43 instance, the principles for perception, attention and thinking in cognitive psychology must be appropriately taken into  
44 consideration in exploring axiomatic methods [14,16]. Moreover, bi-directional recall between neurons can help to define  
45 reasonable mappings between the extension and intension parts of a concept [2]. Concept learning in the machine level is  
46 to be studied in computer science and information science. More attention has been attracted on this aspect because many  
47 kinds of effective methods [1,14,35,49,50] were developed to learn concepts from a given clue. In fact, concept learning in  
48 the abstract, brain and machine levels are relatively independent and closely related to one another. That is to say, on one  
49 hand, each of them can be researched independently. On the other hand, results from any one of them are beneficial to the  
50 better understanding of the other two. In our opinion, only by considering these three aspects in a unified framework can  
51 we have a comprehensive understanding of concept learning. The current work has an interest in the study of three-way  
52 concept learning from the abstract and machine levels.

53 Granular computing [63] has emerged as a unified and coherent platform of constructing, describing, and processing  
54 information granules. Currently, all kinds of models have been designed for information granules [3,4,26–30,36,45,54]. Gen-  
55 erally speaking, the collection of information granules induced by a (resemblance, proximity, functional, etc.) relation can  
56 form a single granularity of the universe of discourse. In many practical applications, however, multiple granularities (of-  
57 ten termed as multi-granularity) are also needed for problem-solving. For example, in a classification problem with several  
58 experts, it is a common situation that different experts have different views on dividing samples into classes. Under such  
59 a circumstance, each expert may give an independent granularity of the samples according to his or her personal pref-  
60 erence. Then the final classification result can be obtained by effectively combining the multiple granularities from these  
61 experts. In fact, the multi-granularity view has been widely used in rough set theory. For instance, considering that multiple  
62 granularities will be generated in multi-scale datasets [47], Wu and Leung [48] studied how to select optimal granularity  
63 for optimization of the granulated information. Optimal granularity selection was also investigated from the viewpoint of  
64 local approaches [40]. Liang et al. [20] adopted the multi-granularity view to accelerate the speed of finding an approxi-  
65 mate reduct. Based on multi-granularity, Qian et al. [33,34] put forward two novel rough set models (i.e. pessimistic and  
66 optimistic multi-granulation rough sets) for information fusion, and they designed a classifier based on these two kinds of  
67 multi-granulation rough sets. Moreover, multi-granularity has further been integrated into neighborhood-based, tolerance,  
68 covering and fuzzy rough set models [11,21,22] for complex information fusion. Also, the classical and generalized rough set  
69 models based on multi-granularity have been compared and connected with other theories such as formal concept analy-

70 sis and cost-sensitive classification [17,39,52,62]. In this paper, multi-granularity will be used to discuss three-way concept  
71 learning.

72 The cognitive viewpoint has been commonly adopted in the study of concept learning [2,14,16,49,50,57,66] because it has  
73 a wide application background in simulating intelligence behaviors of the brain including thinking, learning and reasoning.  
74 For this reason, this idea will also be incorporated into three-way concept learning in this paper. In summary, three-way  
75 cognitive concept learning via multi-granularity deserves to be studied, which is the main issue that our current work  
76 focuses on. Specifically, we propose an axiomatic approach to describe three-way concepts based on multi-granularity, es-  
77 tablish a three-way cognitive computing system for finding composite three-way cognitive concepts, and simulate cognitive  
78 processes for learning three-way concepts from a given clue. Besides, some numerical experiments are conducted to assess  
79 the performance of the proposed learning methods. The main contribution of this paper is to reveal the essential idea of  
80 three-way concepts in solving decision-making problems from the aspect of cognition, meaning that we will clarify which  
81 properties of three-way concepts are intrinsic.

82 The remainder of this paper is organized as follows. Section 2 analyzes cognitive mechanism of forming three-way con-  
83 cepts based on multi-granularity and three-way-decision-making principles. Moreover, the notions of three-way cognitive  
84 operators, three-way cognitive concepts and three-way granular concepts are proposed. Some important properties are also  
85 discussed. Section 3 designs a three-way cognitive computing system which is in fact a dynamic process to update three-way  
86 granular concepts. Section 4 employs the idea of set approximation to simulate cognitive processes for learning three-way  
87 cognitive concepts from a given clue. Section 5 conducts some numerical experiments to evaluate the performance of the  
88 proposed learning methods. The paper is then concluded with a brief summary and an outlook for further research.

## 89 2. Cognitive mechanism of forming three-way concepts

90 In this section, we analyze cognitive mechanism of forming three-way concepts based on multi-granularity and three-  
91 way-decision-making principles. Throughout the paper, we denote by  $U$  a nonempty object set, i.e., the universe of discourse,  
92 and  $A$  an attribute set.

### 93 2.1. Basic notions

94 We first introduce the notions of three-way decisions, three-way quotient set and its power set.

95 In accordance with the notations in rough set theory [25], we still call the sets inducing three-way decisions (i.e., accep-  
96 tance, rejection and non-commitment [60]) as positive, negative and boundary regions [5,10,64,69]. Moreover, the positive,  
97 negative and boundary regions are not distinguished from their respective three-way decisions in the subsequent discus-  
98 sions.

99 For  $x \in U$  and  $A_i \subseteq A$ , let  $f_{A_i}(x)$  be an evaluation function associated to  $A_i$ . Then, given two parameters  $\alpha$  and  $\beta$  with  
100  $\beta < \alpha$ , the positive, negative and boundary regions can be formalized as follows:

- 101 (i) positive region:  $X_i = \{x \in U \mid f_{A_i}(x) \geq \alpha\}$ ,  
102 (ii) negative region:  $Y_i = \{x \in U \mid f_{A_i}(x) \leq \beta\}$ ,  
103 (iii) boundary region:  $Z_i = U - X_i - Y_i$ .

104 It should be pointed out that the evaluation function  $f_{A_i}$  can be defined according to the practical background of the  
105 problem to be solved. In addition, the assignment of values to the parameters  $\alpha$  and  $\beta$  is performed by an expert based on  
106 his or her experience in the field that the problem belongs to.

107 Furthermore, we say that  $X_i$ ,  $Y_i$  and  $Z_i$  are three-way decisions induced by  $A_i$  with the help of  $\alpha$  and  $\beta$ . In fact, from  
108 granular computing, three-way decisions  $X_i$ ,  $Y_i$  and  $Z_i$  form a granularity of  $U$  by eliminating empty sets. Note that these  
109 three-way decisions satisfy  $X_i \cup Y_i \cup Z_i = U$ . So, it is sufficient to describe three-way decisions by any two of them. Here-  
110 inafter, we choose  $(X_i, Y_i)$  to represent three-way decisions when no confusion is caused.

111 Let  $S$  be an index set. Suppose that  $(X_i, Y_i)$  ( $i \in S$ ) are a series of three-way decisions induced by multiple subsets  $A_i$  ( $i \in$   
112  $S$ ) of  $A$ . Then, for each  $i \in S$ , three-way decisions  $(X_i, Y_i)$  can form a granularity of  $U$  by eliminating empty sets. Therefore,  
113  $(X_i, Y_i)$  ( $i \in S$ ) can be viewed as a result of multi-granularity of  $U$ .

114 Moreover, if the multiple subsets  $A_i$  ( $i \in S$ ) constitute a partition of  $A$ , then  $\mathcal{Q}(A) = \{A_i \mid i \in S\}$  is called a three-way  
115 quotient set of  $A$ . For convenience, we denote the power set of  $\mathcal{Q}(A)$  by  $2^{\mathcal{Q}(A)}$ . Here, every  $B \in 2^{\mathcal{Q}(A)}$  can be considered as a  
116 group of knowledge jointly inducing three-way decisions.

### 117 2.2. Three-way cognitive operators induced by multi-granularity and three-way-decision-making principles

118 For two three-way decisions  $(X_i, Y_i)$  and  $(X_j, Y_j)$  of  $U$ , if  $X_i \subseteq X_j$  and  $Y_i \subseteq Y_j$ , then  $(X_j, Y_j)$  is said to be more effective than  
119  $(X_i, Y_i)$ , which we denote by  $(X_i, Y_i) \leq (X_j, Y_j)$ . Moreover, if  $(X_i, Y_i) \leq (X_j, Y_j)$ , we also say that  $(X_i, Y_i)$  is decision-consistent with  
120 respect to  $(X_j, Y_j)$ .

121 The set of three-way decisions, induced by multi-granularity of  $U$ , is denoted by  $\mathcal{T}(U)$ . Furthermore, the intersection and  
122 union in  $\mathcal{T}(U)$  are respectively defined as

$$\begin{aligned} (X_i, Y_i) \cap (X_j, Y_j) &= (X_i \cap X_j, Y_i \cap Y_j), \\ (X_i, Y_i) \cup (X_j, Y_j) &= (X_i \cup X_j, Y_i \cup Y_j). \end{aligned}$$

**Table 1**

A reviewing dataset.

Manuscript	Domain 1		Domain 2		Domain 3		
	Reviewer $r_1$	Reviewer $r_2$	Reviewer $r_3$	Reviewer $r_4$	Reviewer $r_5$	Reviewer $r_6$	Reviewer $r_7$
$x_1$	Accept	Accept					
$x_2$	Accept	Reject					
$x_3$	Reject	Accept					
$x_4$			Accept	Reject			
$x_5$			Accept	Accept			
$x_6$			Reject	Reject			
$x_7$					Accept	Reject	Accept
$x_8$					Accept	Accept	Accept
$x_9$					Reject	Reject	Accept

123 Now, we discuss what the mappings  $\mathcal{H} : 2^{\mathcal{Q}(A)} \rightarrow \mathcal{T}(U)$  and  $\mathcal{L} : \mathcal{T}(U) \rightarrow 2^{\mathcal{Q}(A)}$  need to obey when they are used to form  
 124 three-way concepts?

125 **Three-way-decision-making principle I:** According to sequential or dynamic three-way decisions [9,18,61], the more  
 126 knowledge we use to induce three-way decisions, the more effective the induced three-way decisions are. From this princi-  
 127 ple, we have

$$B_i \subseteq B_j \Rightarrow \mathcal{H}(B_i) \preceq \mathcal{H}(B_j). \quad (1)$$

128 **Three-way-decision-making principle II:** Three-way decisions made by the whole group are less effective than or as  
 129 effective as the combination of those made by its sub-groups. From this principle, we have

$$\mathcal{H}(B_i \cup B_j) \preceq \mathcal{H}(B_i) \cup \mathcal{H}(B_j). \quad (2)$$

130 **Three-way-decision-making principle III:** Whether or not the knowledge is selected depends on how decision-  
 131 consistent its induced three-way decisions are with respect to the target three-way decisions  $(X, Y)$ . From this principle,  
 132 we obtain

$$\mathcal{L}(X, Y) = \{A_i \in \mathcal{Q}(A) \mid \mathcal{H}(\{A_i\}) \preceq (X, Y)\}. \quad (3)$$

133 In what follows, Eqs. (1)–(3) are used as conditions to define three-way cognitive operators based on the mappings  $\mathcal{H}$   
 134 and  $\mathcal{L}$ .

135 **Definition 1.** Given two mappings  $\mathcal{H} : 2^{\mathcal{Q}(A)} \rightarrow \mathcal{T}(U)$  and  $\mathcal{L} : \mathcal{T}(U) \rightarrow 2^{\mathcal{Q}(A)}$ , if for any  $B_i, B_j \in 2^{\mathcal{Q}(A)}$  and  $(X, Y) \in \mathcal{T}(U)$ , the  
 136 following properties hold:

- 137 (i)  $B_i \subseteq B_j \Rightarrow \mathcal{H}(B_i) \preceq \mathcal{H}(B_j)$ ,  
 138 (ii)  $\mathcal{H}(B_i \cup B_j) \preceq \mathcal{H}(B_i) \cup \mathcal{H}(B_j)$ ,  
 139 (iii)  $\mathcal{L}(X, Y) = \{A_i \in \mathcal{Q}(A) \mid \mathcal{H}(\{A_i\}) \preceq (X, Y)\}$ , then  $\mathcal{H}$  and  $\mathcal{L}$  are called three-way cognitive operators.

140 Note that the reason of calling  $\mathcal{H}$  and  $\mathcal{L}$  as three-way cognitive operators is as follows:

- 141 (1) both  $\mathcal{H}$  and  $\mathcal{L}$  involve three-way decisions, i.e., the co-domain of  $\mathcal{H}$  and the domain of  $\mathcal{L}$ ;  
 142 (2)  $\mathcal{H}$  and  $\mathcal{L}$  can be jointly used to form concepts, i.e., recognition of concepts.

143 In addition, it should be pointed out that the properties (i), (ii) and (iii) used to define three-way cognitive operators are  
 144 from three-way-decision-making principles I, II and III, respectively. In other words, there are explicit semantics for these  
 145 properties. In fact, the properties (i)–(iii) are very important because they can be jointly used as axioms to characterize  
 146 three-way concepts. So, these properties can be considered as the intrinsic ones of characterizing three-way concepts.

147 **Remark 1.** For three-way cognitive operators  $\mathcal{H}$  and  $\mathcal{L}$ , we say that three-way decisions  $(X, Y)$  induced by a nonempty set  
 148  $B \in 2^{\mathcal{Q}(A)}$  (i.e.  $\mathcal{H}(B) = (X, Y)$ ) are trivial if  $X$  and  $Y$  are empty simultaneously. Hereinafter, three-way decisions induced by  
 149 any nonempty set  $B \in 2^{\mathcal{Q}(A)}$  are assumed to be not trivial.

150 **Remark 2.** For three-way decisions  $(X_i, Y_i)$  and  $(X_j, Y_j)$  induced by two nonempty sets  $B_i, B_j \in 2^{\mathcal{Q}(A)}$ , if  $X_i \cap Y_j = \emptyset$  and  $Y_i \cap$   
 151  $X_j = \emptyset$ , we say that  $(X_i, Y_i)$  and  $(X_j, Y_j)$  are contradictory with each other. In the rest of this paper, three-way decisions  
 152 induced by any two nonempty sets  $B_i, B_j \in 2^{\mathcal{Q}(A)}$  are assumed to be contradictory with each other.

153 For conciseness, we also write  $\mathcal{H}(\{A_i\})$  as  $\mathcal{H}(A_i)$  when no confusion is caused.

154 **Example 1.** Table 1 depicts a dataset of nine manuscripts evaluated by seven reviewers who are from three domains. That  
 155 is, the first two reviewers are from Domain 1, the second two from Domain 2, and the remainder from Domain 3. In the  
 156 table, null value in the cross of a row and a column means that the manuscript in this row was not assigned to be evaluated  
 157 by the reviewer from this column. It is easy to observe that the manuscripts 1–3 fall into the first domain, the manuscripts  
 158 4–6 the second domain, and the manuscripts 7–9 the third domain.

159 Let  $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$  be the set of nine manuscripts, and  $A = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$  be the set of seven  
 160 reviewers. Then,  $A_1 = \{r_1, r_2\}$ ,  $A_2 = \{r_3, r_4\}$  and  $A_3 = \{r_5, r_6, r_7\}$  are just the reviewers from three domains, respectively. Since



161  $A_1, A_2$  and  $A_3$  form a partition of  $A$ , then  $\mathcal{Q}(A) = \{A_1, A_2, A_3\}$  is a three-way quotient set of  $A$ . Moreover, its power set is as  
162 follows:

$$2^{\mathcal{Q}(A)} = \{\emptyset, \{A_1\}, \{A_2\}, \{A_3\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}, \{A_1, A_2, A_3\}\}.$$

163 Take  $\alpha = \frac{2}{3}$  and  $\beta = \frac{1}{3}$ . Suppose the evaluation function  $f_{B_i}(x)$  ( $B_i \in 2^{\mathcal{Q}(A)}$ ,  $x \in U$ ) is the ratio of the number of Accepts  
164 given to  $x$  to that of reviewers assigned to  $x$  under the columns  $\cup B_i$ . Note that  $x$  will be put into the boundary region  
165 directly if the total number of Accepts and Rejects given to  $x$  under the columns  $\cup B_i$  is less than or equal to 1. In other  
166 words, a non-commitment decision will be made to  $x$  if it does not receive enough evaluations. Then we can generate the  
167 following three-way decisions:

- 168 • positive region induced by  $\{A_1\}$ :  $X_1 = \{x \in U \mid f_{\{A_1\}}(x) \geq \alpha\} = \{x_1\}$ ,
- 169 • negative region induced by  $\{A_1\}$ :  $Y_1 = \{x \in U \mid f_{\{A_1\}}(x) \leq \beta\} = \emptyset$ ,
- 170 • positive region induced by  $\{A_2\}$ :  $X_2 = \{x \in U \mid f_{\{A_2\}}(x) \geq \alpha\} = \{x_5\}$ ,
- 171 • negative region induced by  $\{A_2\}$ :  $Y_2 = \{x \in U \mid f_{\{A_2\}}(x) \leq \beta\} = \{x_6\}$ ,
- 172 • positive region induced by  $\{A_3\}$ :  $X_3 = \{x \in U \mid f_{\{A_3\}}(x) \geq \alpha\} = \{x_7, x_8\}$ ,
- 173 • negative region induced by  $\{A_3\}$ :  $Y_3 = \{x \in U \mid f_{\{A_3\}}(x) \leq \beta\} = \{x_9\}$ ,
- 174 • positive region induced by  $\{A_1, A_2\}$ :  $X_4 = \{x \in U \mid f_{\{A_1, A_2\}}(x) \geq \alpha\} = \{x_1, x_5\}$ ,
- 175 • negative region induced by  $\{A_1, A_2\}$ :  $Y_4 = \{x \in U \mid f_{\{A_1, A_2\}}(x) \leq \beta\} = \{x_6\}$ ,
- 176 • positive region induced by  $\{A_1, A_3\}$ :  $X_5 = \{x \in U \mid f_{\{A_1, A_3\}}(x) \geq \alpha\} = \{x_1, x_7, x_8\}$ ,
- 177 • negative region induced by  $\{A_1, A_3\}$ :  $Y_5 = \{x \in U \mid f_{\{A_1, A_3\}}(x) \leq \beta\} = \{x_9\}$ ,
- 178 • positive region induced by  $\{A_2, A_3\}$ :  $X_6 = \{x \in U \mid f_{\{A_2, A_3\}}(x) \geq \alpha\} = \{x_5, x_7, x_8\}$ ,
- 179 • negative region induced by  $\{A_2, A_3\}$ :  $Y_6 = \{x \in U \mid f_{\{A_2, A_3\}}(x) \leq \beta\} = \{x_6, x_9\}$ ,
- 180 • positive region induced by  $\{A_1, A_2, A_3\}$ :  $X_7 = \{x \in U \mid f_{\{A_1, A_2, A_3\}}(x) \geq \alpha\} = \{x_1, x_5, x_7, x_8\}$ ,
- 181 • negative region induced by  $\{A_1, A_2, A_3\}$ :  $Y_7 = \{x \in U \mid f_{\{A_1, A_2, A_3\}}(x) \leq \beta\} = \{x_6, x_9\}$ .

182 Thus,  $\mathcal{T}(U) = \{(\emptyset, \emptyset), (X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4), (X_5, Y_5), (X_6, Y_6), (X_7, Y_7)\}$  is the set of three-way decisions in-  
183 duced by multi-granularity of  $U$ . It should be pointed out that the trivial three-way decisions  $(\emptyset, \emptyset)$  is forcibly included  
184 in  $\mathcal{T}(U)$  in order to establish the following mappings:

$$\begin{aligned} \mathcal{H}: \quad & \emptyset \mapsto (\emptyset, \emptyset), \{A_1\} \mapsto (X_1, Y_1), \{A_2\} \mapsto (X_2, Y_2), \{A_3\} \mapsto (X_3, Y_3), \\ & \{A_1, A_2\} \mapsto (X_4, Y_4), \{A_1, A_3\} \mapsto (X_5, Y_5), \{A_2, A_3\} \mapsto (X_6, Y_6), \{A_1, A_2, A_3\} \mapsto (X_7, Y_7) \end{aligned}$$

185 and

$$\begin{aligned} \mathcal{L}: \quad & (\emptyset, \emptyset) \mapsto \emptyset, \quad (X_1, Y_1) \mapsto \{A_1\}, \quad (X_2, Y_2) \mapsto \{A_2\}, \quad (X_3, Y_3) \mapsto \{A_3\}, \\ & (X_4, Y_4) \mapsto \{A_1, A_2\}, \quad (X_5, Y_5) \mapsto \{A_1, A_3\}, \quad (X_6, Y_6) \mapsto \{A_2, A_3\}, \quad (X_7, Y_7) \mapsto \{A_1, A_2, A_3\}. \end{aligned}$$

186 Then, based on Definition 1, the mappings  $\mathcal{H}$  and  $\mathcal{L}$  are three-way cognitive operators.

187 **Proposition 1.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be three-way cognitive operators. Then for any nonempty set  $B \in 2^{\mathcal{Q}(A)}$ , we have

$$\mathcal{H}(B) = \bigcup_{A_i \in B} \mathcal{H}(A_i). \quad (4)$$

188 **Proof.** To complete the proof, it is sufficient to show  $\mathcal{H}(\{A_i, A_j\}) = \mathcal{H}(A_i) \cup \mathcal{H}(A_j)$ , where  $A_i, A_j \in B$ . By Eq. (1), we have  
189  $\mathcal{H}(A_i) \cup \mathcal{H}(A_j) \subseteq \mathcal{H}(\{A_i, A_j\})$  due to  $\mathcal{H}(A_i) \subseteq \mathcal{H}(\{A_i, A_j\})$  and  $\mathcal{H}(A_j) \subseteq \mathcal{H}(\{A_i, A_j\})$ . By combining  $\mathcal{H}(A_i) \cup \mathcal{H}(A_j) \subseteq \mathcal{H}(\{A_i, A_j\})$   
190 with Eq. (2), we conclude  $\mathcal{H}(\{A_i, A_j\}) = \mathcal{H}(A_i) \cup \mathcal{H}(A_j)$ .  $\square$

191 **Proposition 2.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be three-way cognitive operators. For any  $B \in 2^{\mathcal{Q}(A)}$  and  $(X, Y), (X_i, Y_i), (X_j, Y_j) \in \mathcal{T}(U)$ , we have the  
192 following properties:

$$B \subseteq \mathcal{L}\mathcal{H}(B); \quad (5)$$

$$\mathcal{H}\mathcal{L}(X, Y) \subseteq (X, Y); \quad (6)$$

$$(X_i, Y_i) \subseteq (X_j, Y_j) \Rightarrow \mathcal{L}(X_i, Y_i) \subseteq \mathcal{L}(X_j, Y_j), \quad (7)$$

195 where  $\mathcal{H}\mathcal{L}(\bullet)$  and  $\mathcal{L}\mathcal{H}(\bullet)$  represent the composite mappings  $\mathcal{H}(\mathcal{L}(\bullet))$  and  $\mathcal{L}(\mathcal{H}(\bullet))$ , respectively.

196 **Proof.** Firstly, we prove Eq. (5). For any  $A_i \in B$ , we have  $\mathcal{H}(A_i) \subseteq \mathcal{H}(B)$  according to Eq. (1). Based on Eq. (3), we obtain  
197  $A_i \in \mathcal{L}\mathcal{H}(B)$ . As a result,  $B \subseteq \mathcal{L}\mathcal{H}(B)$  is true.

198 Secondly, we prove Eq. (6). For any  $A_i \in \mathcal{L}(X, Y)$ , by Eq. (3), we get  $\mathcal{H}(A_i) \subseteq (X, Y)$ . It can be seen from Proposition 1 that  
199  $\mathcal{H}\mathcal{L}(X, Y) = \bigcup_{A_i \in \mathcal{L}(X, Y)} \mathcal{H}(A_i) \subseteq (X, Y)$ .

200 Finally, we prove Eq. (7). Suppose  $(X_i, Y_i) \subseteq (X_j, Y_j)$ . Then, for any  $A_i \in \mathcal{L}(X_i, Y_i)$ , we know from Eq. (3) that  $\mathcal{H}(A_i) \subseteq (X_i, Y_i)$   
201 is true. Moreover, we have  $\mathcal{H}(A_i) \subseteq (X_j, Y_j)$ . Consequently,  $A_i \in \mathcal{L}(X_j, Y_j)$  is proved.  $\square$

202 It deserves to point out that based on Eqs. (1), (5), (6) and (7), the pair  $(\mathcal{H}, \mathcal{L})$  forms an isotone Galois connection  
203 [24,46] between  $2^{\mathcal{Q}(A)}$  and  $\mathcal{T}(U)$ . This means that the mappings  $\mathcal{H}$  and  $\mathcal{L}$  can be jointly used to induce concepts. By the  
204 way, three-way cognitive operators  $\mathcal{H}$  and  $\mathcal{L}$  are completely different from the classical ones [16] which form an antitone  
205 Galois connection between  $2^A$  and  $2^U$ .

206 2.3. Three-way cognitive concepts and three-way granular concepts

207 In this subsection, we put forward the notion of a three-way cognitive concept and discuss information granules for  
208 three-way cognitive concepts.

209 **Definition 2.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be three-way cognitive operators. For  $B \in 2^{\mathcal{Q}(A)}$  and  $(X, Y) \in \mathcal{T}(U)$ , if  $\mathcal{H}(B) = (X, Y)$  and  $\mathcal{L}(X, Y) =$   
210  $B$ , we say that  $\langle (X, Y), B \rangle$  is a three-way concept under the cognitive operators  $\mathcal{H}$  and  $\mathcal{L}$  (or simply a three-way cognitive  
211 concept). In this case,  $(X, Y)$  and  $B$  are called the extent and intent of the three-way cognitive concept  $\langle (X, Y), B \rangle$ , respectively.  
212 Henceforth, the set of all three-way cognitive concepts is denoted by  $\underline{\mathfrak{B}}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$ .

213 Three-way cognitive concepts are different from triadic concepts [13] which consist of extent, intent and modus. The  
214 reasons are as follows:

215 (i) The extent of a three-way cognitive concept is constituted by positive region, negative region and boundary region,  
216 while that of a triadic concept is only a subset of the universe of discourse.

217 (ii) The attribute sets  $A_i (i \in S)$  used to induce basic three-way decisions are pairwise disjoint (see, e.g., those under  
218 Domains 1–3 in Table 1), while the attribute sets under conditions of a triadic context are the same. That is to say, the  
219 relationship between the extent and intent of a three-way cognitive concept is different from that of a triadic concept. More  
220 specifically, the former claims that the intent is considered as an evaluation function to partition the universe of discourse  
221 into positive, negative and boundary regions for generating extent, while the latter emphasizes that each object in extent  
222 has all the attributes in intent under every condition in modus.

223 The infimum ( $\wedge$ ) and supremum ( $\vee$ ) among three-way cognitive concepts  $\underline{\mathfrak{B}}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$  are respectively defined  
224 as:

$$\begin{aligned} \langle (X_i, Y_i), B_i \rangle \wedge \langle (X_j, Y_j), B_j \rangle &= \langle \mathcal{H}\mathcal{L}((X_i, Y_i) \cap (X_j, Y_j)), B_i \cap B_j \rangle, \\ \langle (X_i, Y_i), B_i \rangle \vee \langle (X_j, Y_j), B_j \rangle &= \langle (X_i, Y_i) \cup (X_j, Y_j), \mathcal{L}\mathcal{H}(B_i \cup B_j) \rangle. \end{aligned} \quad (8)$$

225 **Example 2** (Continued with Example 1). Based on Definition 2, we can obtain the following three-way cognitive concepts  
226 for Example 1:

$$\begin{aligned} &\langle (\{x_1, x_5, x_7, x_8\}, \{x_6, x_9\}), \mathcal{Q}(A) \rangle, \quad \langle (\emptyset, \emptyset), \emptyset \rangle, \quad \langle (\{x_1, x_5\}, \{x_6\}), \{A_1, A_2\} \rangle, \quad \langle (\{x_1, x_7, x_8\}, \{x_9\}), \{A_1, A_3\} \rangle, \\ &\langle (\{x_5, x_7, x_8\}, \{x_6, x_9\}), \{A_2, A_3\} \rangle, \quad \langle (\{x_1\}, \emptyset), \{A_1\} \rangle, \quad \langle (\{x_5\}, \{x_6\}), \{A_2\} \rangle, \quad \langle (\{x_7, x_8\}, \{x_9\}), \{A_3\} \rangle, \end{aligned}$$

227 where  $A_1 = \{r_1, r_2\}$ ,  $A_2 = \{r_3, r_4\}$  and  $A_3 = \{r_5, r_6, r_7\}$ .

228 **Definition 3.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be three-way cognitive operators. Then  $\mathcal{H}^G = \{\{A_i\} \rightarrow \mathcal{H}(A_i) \mid A_i \in \mathcal{Q}(A)\}$  is called information  
229 granules of  $\mathcal{H}$ .

230 According to Eq. (4), the information granules  $\mathcal{H}^G$  can be used to generate the mapping  $\mathcal{H}$ .

231 **Proposition 3.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be three-way cognitive operators. Then for any  $B \in 2^{\mathcal{Q}(A)}$ ,  $\langle \mathcal{H}(B), \mathcal{L}\mathcal{H}(B) \rangle$  is a three-way cognitive  
232 concept.

233 **Proof.** The conclusion is immediate from Definition 2 and Proposition 2.  $\square$

234 Proposition 3 is further used to define the notion of a three-way granular concept by taking  $B = \{A_i\}$  and the formaliza-  
235 tion is given below.

236 **Definition 4.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be three-way cognitive operators. Then for any singleton set  $\{A_i\} \in 2^{\mathcal{Q}(A)}$ , we say that  
237  $\langle \mathcal{H}(A_i), \mathcal{L}\mathcal{H}(A_i) \rangle$  is a three-way granular concept.

238 **Proposition 4.** Let  $\mathcal{H}$  and  $\mathcal{L}$  be three-way cognitive operators. Then for any  $\langle (X, Y), B \rangle \in \underline{\mathfrak{B}}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$ , we have

$$\langle (X, Y), B \rangle = \bigvee_{A_i \in B} \langle \mathcal{H}(A_i), \mathcal{L}\mathcal{H}(A_i) \rangle. \quad (9)$$

239 **Proof.** The conclusion can be obtained directly from Eqs. (4) and (8).  $\square$

240 Proposition 4 indicates that any three-way cognitive concept can be induced by integrating three-way granular con-  
241 cepts. Thus, from granular computing, three-way granular concepts can be considered as the information granules of  
242  $\underline{\mathfrak{B}}(2^{\mathcal{Q}(A)}, \mathcal{T}(U), \mathcal{H}, \mathcal{L})$ . Hereinafter, we denote the collection of the information granules by  $G_{\mathcal{H}\mathcal{L}}$ . That is,

$$G_{\mathcal{H}\mathcal{L}} = \{\langle \mathcal{H}(A_i), \mathcal{L}\mathcal{H}(A_i) \rangle \mid A_i \in \mathcal{Q}(A)\}. \quad (10)$$

243 **Example 3** (Continued with Example 2). Note that the following equations hold for Example 2:

$$\begin{aligned} \mathcal{H}(A_1) &= (\{x_1\}, \emptyset), & \mathcal{L}\mathcal{H}(A_1) &= \mathcal{L}(\{x_1\}, \emptyset) = \{A_1\}, \\ \mathcal{H}(A_2) &= (\{x_5\}, \{x_6\}), & \mathcal{L}\mathcal{H}(A_2) &= \mathcal{L}(\{x_5\}, \{x_6\}) = \{A_2\}, \\ \mathcal{H}(A_3) &= (\{x_7, x_8\}, \{x_9\}), & \mathcal{L}\mathcal{H}(A_3) &= \mathcal{L}(\{x_7, x_8\}, \{x_9\}) = \{A_3\}, \end{aligned}$$

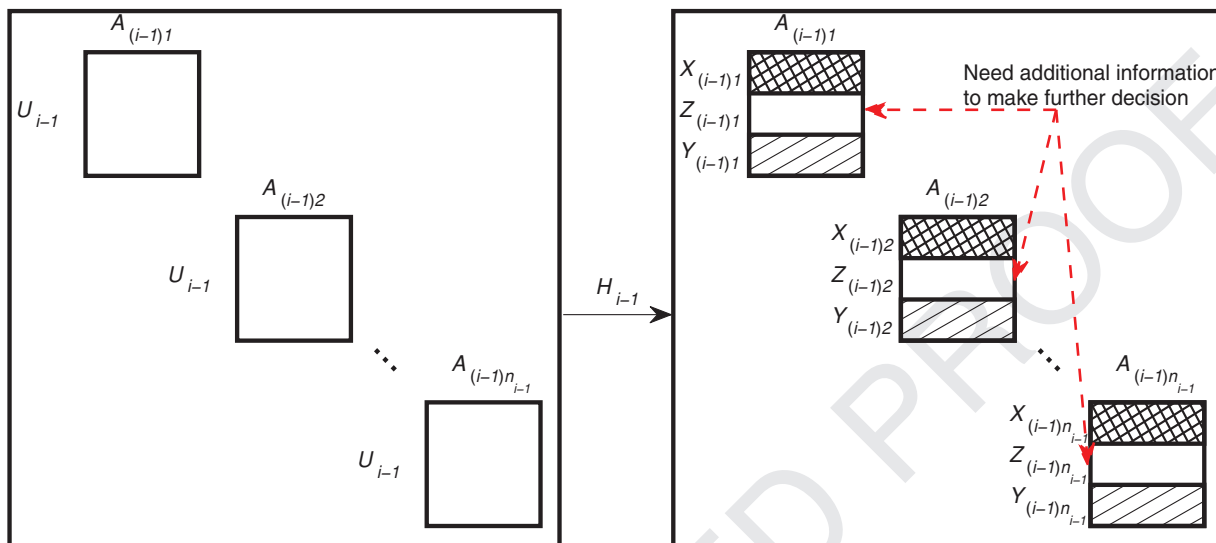


Fig. 1. The graph of  $\mathcal{H}_{i-1}$ .

244 where  $A_1 = \{r_1, r_2\}$ ,  $A_2 = \{r_3, r_4\}$  and  $A_3 = \{r_5, r_6, r_7\}$ . Thus, based on Definition 4, we know that  $\langle\langle\{x_1\}, \emptyset, \{A_1\}\rangle\rangle$ ,  $\langle\langle\{x_5\}, \{x_6\}, \{A_2\}\rangle\rangle$  and  $\langle\langle\{x_7, x_8\}, \{x_9\}, \{A_3\}\rangle\rangle$  are three-way granular concepts. That is,

$$G_{\mathcal{H}\mathcal{L}} = \{\langle\langle\{x_1\}, \emptyset, \{A_1\}\rangle\rangle, \langle\langle\{x_5\}, \{x_6\}, \{A_2\}\rangle\rangle, \langle\langle\{x_7, x_8\}, \{x_9\}, \{A_3\}\rangle\rangle\},$$

246 which can be further used to induce other three-way cognitive concepts.

### 247 3. Three-way cognitive computing system

248 From cognitive computing, concepts should be updated to simulate intelligence behaviors of the brain when information  
 249 is updated periodically. For instance, in Example 1, nine manuscripts were evaluated by seven reviewers. As time goes by,  
 250 on one hand, new manuscripts will arrive. On the other hand, those falling into the boundary regions need to be evaluated  
 251 by inviting additional reviewers. In this case, it is necessary to update three-way granular concepts for supporting a further  
 252 decision of the manuscripts with non-commitment decisions.

253 Motivated by the above problem, we propose in this section a three-way cognitive computing system to update three-  
 254 way granular concepts as objects and/or attributes increase in batches. Before embarking on this issue, we introduce some  
 255 notations.

256 To facilitate our subsequent discussion,  $n$  attribute sets  $A_1, A_2, \dots, A_n$  with  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n$  are denoted by  $\{A_t | t \in$   
 257  $S\}$ , where  $S = \{1, 2, \dots, n\}$ . Similarly,  $n$  object sets  $U_1, U_2, \dots, U_n$  with  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n$  are denoted by  $\{U_t | t \in S\}$ . Let  
 258  $\Delta A_{i-1} = A_i - A_{i-1}$  and  $\Delta U_{i-1} = U_i - U_{i-1}$ . Moreover, for any  $i \in S$ , we denote by  $2^{\mathcal{Q}(A_i)}$  the power set of three-way quotient  
 259 set of  $A_i$ , and by  $\mathcal{T}(U_i)$  the set of three-way decisions of  $U_i$ .

260 For any  $A_{i-1s} \in \mathcal{Q}(A_{i-1})$ , if there exists  $A_{it} \in \mathcal{Q}(A_i)$  such that  $A_{i-1s} \subseteq A_{it}$ , then  $\mathcal{Q}(A_i)$  is called a generalization of  $\mathcal{Q}(A_{i-1})$   
 261 or equivalently  $\mathcal{Q}(A_{i-1})$  is a specification of  $\mathcal{Q}(A_i)$ , where  $i-1, s, i$  and  $t$  are subscript indices. We denote this generaliza-  
 262 tion/specification relation by  $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$ . Such a relation is easy to be understood in the real world. For instance, in  
 263 Example 1, new invited reviewers for evaluating the manuscripts falling into the boundary regions must be from Domain 1,  
 264 Domain 2, Domain 3 or a new domain.

265 Suppose that

$$\mathcal{H}_{i-1} : 2^{\mathcal{Q}(A_{i-1})} \rightarrow \mathcal{T}(U_{i-1}) \tag{11}$$

266 is a mapping from  $2^{\mathcal{Q}(A_{i-1})}$  to  $\mathcal{T}(U_{i-1})$  and

$$\mathcal{H}_{\Delta U_{i-1}} : 2^{\mathcal{Q}(A_{i-1})} \rightarrow \mathcal{T}(\Delta U_{i-1}) \tag{12}$$

267 is a mapping from  $2^{\mathcal{Q}(A_{i-1})}$  to  $\mathcal{T}(\Delta U_{i-1})$ . In what follows,  $\mathcal{H}_{i-1}$  and  $\mathcal{H}_{\Delta U_{i-1}}$  are further explained by graphs for better under-  
 268 standing of their decision-making mechanisms.

269 First of all, let us begin with the mapping  $\mathcal{H}_{i-1}$ . Note that  $\mathcal{H}_{i-1}$  can be completely determined by its information granules  
 270  $\mathcal{H}_{i-1}^G$ . So, Fig. 1 shows the graph of  $\mathcal{H}_{i-1}$ , where each attribute set  $A_{i-1j}$  ( $j = 1, 2, \dots, n_{i-1}$ ) partitions  $U_{i-1}$  into three-way  
 271 decisions  $X_{i-1j}$ ,  $Y_{i-1j}$  and  $Z_{i-1j}$ . Obviously, the boundary regions  $Z_{i-1j}$  ( $j = 1, 2, \dots, n_{i-1}$ ) need additional information to make  
 272 decision.

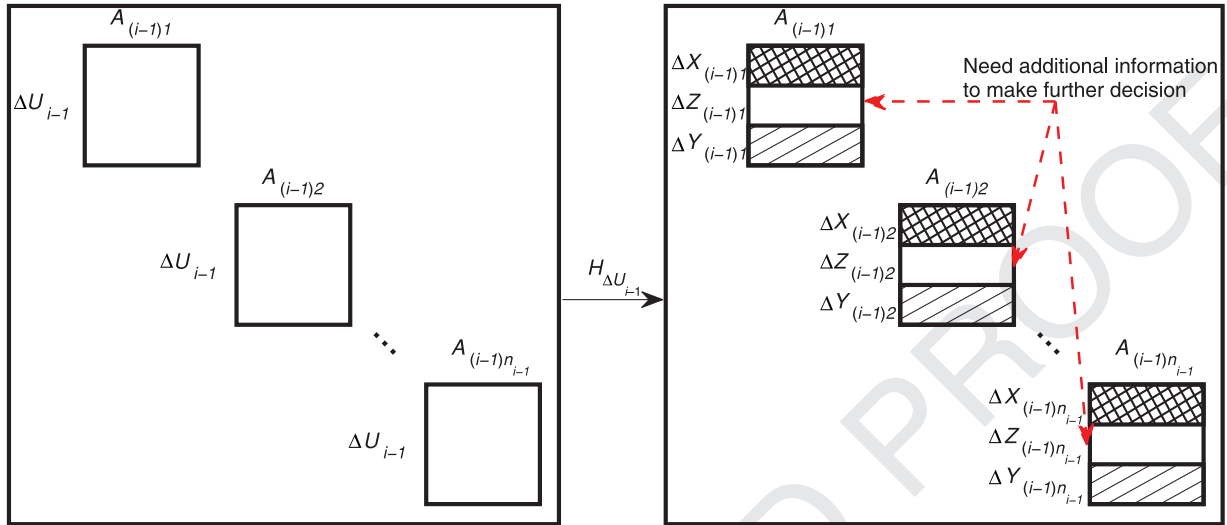


Fig. 2. The graph of  $\mathcal{H}_{\Delta U_{i-1}}$ .

273 Then, we continue to analyze the mapping  $\mathcal{H}_{\Delta U_{i-1}}$ . Like the case of  $\mathcal{H}_{i-1}$ , the information granules  $\mathcal{H}_{\Delta U_{i-1}}^G$  can determine  
 274  $\mathcal{H}_{\Delta U_{i-1}}$  with certainty. Thus, Fig. 2 in fact depicts the graph of  $\mathcal{H}_{\Delta U_{i-1}}$ , where each attribute set  $A_{i-1j}$  ( $j = 1, 2, \dots, n_{i-1}$ )  
 275 partitions  $\Delta U_{i-1}$  into three-way decisions  $\Delta X_{i-1j}$ ,  $\Delta Y_{i-1j}$  and  $\Delta Z_{i-1j}$ . Undoubtedly, the boundary regions  $\Delta Z_{i-1j}$  ( $j =$   
 276  $1, 2, \dots, n_{i-1}$ ) also need additional information to make decision.

277 Furthermore, suppose that

$$\mathcal{H}_{\Delta A_{i-1}} : 2^{\mathcal{Q}(A_i)} \rightarrow \mathcal{T}(U^*) \tag{13}$$

278 is a mapping from  $2^{\mathcal{Q}(A_i)}$  to  $\mathcal{T}(U^*)$ , where  $\mathcal{Q}(A_{i-1}) \subseteq \mathcal{Q}(A_i)$  and  $U^* = \bigcup_{j=1}^{n_{i-1}} (Z_{i-1j} \cup \Delta Z_{i-1j})$ . The boundary regions  $Z_{i-1j}$  and  
 279  $\Delta Z_{i-1j}$  can be found in Figs. 1 and 2, respectively. In other words, the attribute set  $\Delta A_{i-1}$  is combined with  $A_{i-1}$  for sup-  
 280 porting a further decision to the objects in the boundary regions. This is in accordance with the idea of sequential or  
 281 dynamic three-way decisions [9,18,61]. In addition, the graph of  $\mathcal{H}_{\Delta A_{i-1}}$  is shown in Fig. 3. In the figure, each attribute  
 282 set  $A_{ij}$  ( $j = 1, 2, \dots, n_{i-1}$ ) partitions  $U^*$  into three-way decisions  $(X_{i-1j}^Z \cup \Delta X_{i-1j}^Z, Y_{i-1j}^Z \cup \Delta Y_{i-1j}^Z)$ , while each attribute set  $A_{ij}$   
 283 ( $j = n_{i-1} + 1, \dots, n_i$ ) partitions  $U^*$  into three-way decisions  $(X_{ij}, Y_{ij})$ .

284 Finally, Eqs. (11)–(13) are jointly used to construct a new mapping

$$\mathcal{H}_i : 2^{\mathcal{Q}(A_i)} \rightarrow \mathcal{T}(U_i) \tag{14}$$

285 in which the information granules of  $\mathcal{H}_i$  are defined as

$$\mathcal{H}_i(A_{it}) = \begin{cases} \mathcal{H}_{i-1}(A_{i-1s}) \cup \mathcal{H}_{\Delta U_{i-1}}(A_{i-1s}) \cup \mathcal{H}_{\Delta A_{i-1}}(A_{it}), & \text{if } \exists A_{i-1s} \in \mathcal{Q}(A_{i-1}) \text{ s.t. } A_{i-1s} \subseteq A_{it}, \\ \mathcal{H}_{\Delta A_{i-1}}(A_{it}), & \text{otherwise.} \end{cases} \tag{15}$$

286 Here,  $\mathcal{H}_{\Delta U_{i-1}}(A_{i-1s})$  is set to be empty when  $\Delta U_{i-1} = \emptyset$ , so is  $\mathcal{H}_{\Delta A_{i-1}}(A_{it})$  set when  $\Delta A_{i-1} = \emptyset$ .

287 Fig. 4 shows how to obtain the graph of  $\mathcal{H}_i$  based on those of  $\mathcal{H}_{i-1}$ ,  $\mathcal{H}_{\Delta U_{i-1}}$  and  $\mathcal{H}_{\Delta A_{i-1}}$ . In the figure, each attribute set  
 288  $A_{ij}$  ( $j = 1, 2, \dots, n_{i-1}$ ) partitions  $U_i$  into three-way decisions

$$(X_{i-1j} \cup X_{i-1j}^Z \cup \Delta X_{i-1j} \cup \Delta X_{i-1j}^Z, Y_{i-1j} \cup Y_{i-1j}^Z \cup \Delta Y_{i-1j} \cup \Delta Y_{i-1j}^Z),$$

289 while each attribute set  $A_{ij}$  ( $j = n_{i-1} + 1, \dots, n_i$ ) partitions  $U_i$  into three-way decisions  $(X_{ij}, Y_{ij})$ .

290 **Definition 5.** Let  $A_{i-1}, A_i$  be the attribute sets of  $\{A_t | t \in S\}$ ,  $U_{i-1}, U_i$  be the object sets of  $\{U_t | t \in S\}$  and  $\mathcal{Q}(A_{i-1}) \subseteq \mathcal{Q}(A_i)$ . De-  
 291 note  $\Delta A_{i-1} = A_i - A_{i-1}$  and  $\Delta U_{i-1} = U_i - U_{i-1}$ . Suppose that  $\mathcal{H}_{i-1}, \mathcal{L}_{i-1}$  and  $\mathcal{H}_i, \mathcal{L}_i$  are three-way cognitive operators, where  
 292  $\mathcal{H}_i$  is constructed by  $\mathcal{H}_{i-1}, \mathcal{H}_{\Delta U_{i-1}}$  and  $\mathcal{H}_{\Delta A_{i-1}}$  based on Eq. (14). Then, we say that  $\mathcal{H}_i$  and  $\mathcal{L}_i$  are extended three-way cog-  
 293 nitive operators of  $\mathcal{H}_{i-1}$  and  $\mathcal{L}_{i-1}$  by combining the information  $\mathcal{H}_{\Delta U_{i-1}}$  and  $\mathcal{H}_{\Delta A_{i-1}}$ .

294 **Definition 6.** Let  $A_{i-1}, A_i$  be the attribute sets of  $\{A_t | t \in S\}$ ,  $U_{i-1}, U_i$  be the object sets of  $\{U_t | t \in S\}$  and  $\mathcal{Q}(A_{i-1}) \subseteq \mathcal{Q}(A_i)$ .  
 295 Denote  $\Delta A_{i-1} = A_i - A_{i-1}$  and  $\Delta U_{i-1} = U_i - U_{i-1}$ . Suppose that  $\mathcal{H}_i$  and  $\mathcal{L}_i$  are extended three-way cognitive operators of  $\mathcal{H}_{i-1}$   
 296 and  $\mathcal{L}_{i-1}$  by combining  $\mathcal{H}_{\Delta U_{i-1}}$  and  $\mathcal{H}_{\Delta A_{i-1}}$ . Then, we call  $\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i} = (G_{\mathcal{H}_{i-1} \mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$  a three-way cognitive computing  
 297 state, where  $G_{\mathcal{H}_{i-1} \mathcal{L}_{i-1}}$  is the set of three-way granular concepts under  $\mathcal{H}_{i-1}$  and  $\mathcal{L}_{i-1}$ . Moreover, a collection of three-way  
 298 cognitive computing states, denoted by  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$ , is called a three-way cognitive computing system.



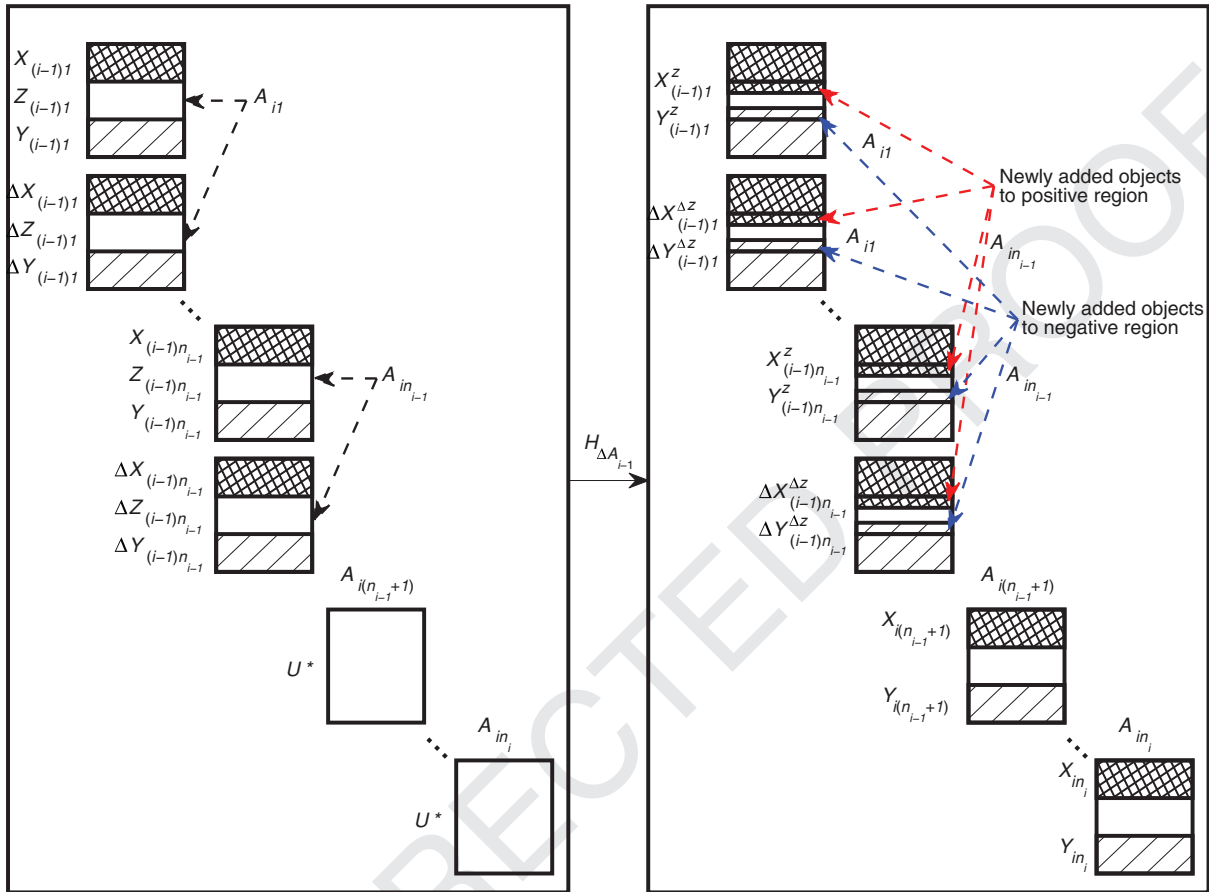


Fig. 3. The graph of  $\mathcal{H}_{\Delta A_{i-1}}$ .

299 Note that the objective of designing a three-way cognitive computing system is to update three-way granular concepts  
 300 as objects and/or attributes increase in batches. This is in accordance with our common sense that in the real world, recog-  
 301 nition of concepts will be gradually improved under the circumstance of information updating until it maintains relative  
 302 stability. In order to achieve this task, it is necessary to analyze the transformation mechanism between three-way granular  
 303 concepts from one three-way cognitive computing state to another.

304 **Proposition 5.** Let  $\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i} = (G_{\mathcal{H}_{i-1} \mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$  be a three-way cognitive computing state and  $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$ . For any  
 305  $A_{it} \in \mathcal{Q}(A_i)$ , if there exists  $A_{i-1s} \in \mathcal{Q}(A_{i-1})$  such that  $A_{i-1s} \subseteq A_{it}$ , we have

$$\mathcal{H}_i(A_{it}) = \mathcal{H}_{i-1}(A_{i-1s}) \cup \mathcal{H}_{\Delta U_{i-1}}(A_{i-1s}) \cup \mathcal{H}_{\Delta A_{i-1}}(A_{it}); \tag{16}$$

306 otherwise,

$$\mathcal{H}_i(A_{it}) = \mathcal{H}_{\Delta A_{i-1}}(A_{it}). \tag{17}$$

307 **Proof.** The proof is immediate from Eqs. (14) and (15).  $\square$

308 **Remark 3.** Based on (iii) of Definition 1, we have

$$\mathcal{L}_i \mathcal{H}_i(A_{it}) = \{A_{is} \in \mathcal{Q}(A_i) \mid \mathcal{H}_i(A_{is}) \preceq \mathcal{H}_i(A_{it})\}. \tag{18}$$

309 Since every  $\mathcal{H}_i(A_{it})$  ( $A_{it} \in \mathcal{Q}(A_i)$ ) can be obtained by Proposition 5, it is easy to compute each  $\mathcal{L}_i \mathcal{H}_i(A_{it})$  according to  
 310 Eqs. (16)–(18).

311 Proposition 5 and Remark 3 can be jointly used to achieve the task of transforming the information granules  $G_{\mathcal{H}_{i-1} \mathcal{L}_{i-1}}$  to  
 312  $G_{\mathcal{H}_i \mathcal{L}_i}$ .

313 Note that for a given three-way cognitive computing system  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$ , all information granules  $G_{\mathcal{H}_1 \mathcal{L}_1}, G_{\mathcal{H}_2 \mathcal{L}_2}, \dots,$   
 314  $G_{\mathcal{H}_n \mathcal{L}_n}$  are unknown in advance. Their sequential computation processes are described below:

- 315 (i) three-way cognitive operators  $\mathcal{H}_1$  and  $\mathcal{L}_1$  are used to compute  $G_{\mathcal{H}_1 \mathcal{L}_1}$  according to Eq. (10),
- 316 (ii) and then recursive strategy is adopted to generate  $G_{\mathcal{H}_2 \mathcal{L}_2}, \dots, G_{\mathcal{H}_n \mathcal{L}_n}$  in sequence based on Proposition 5 and Remark 3.

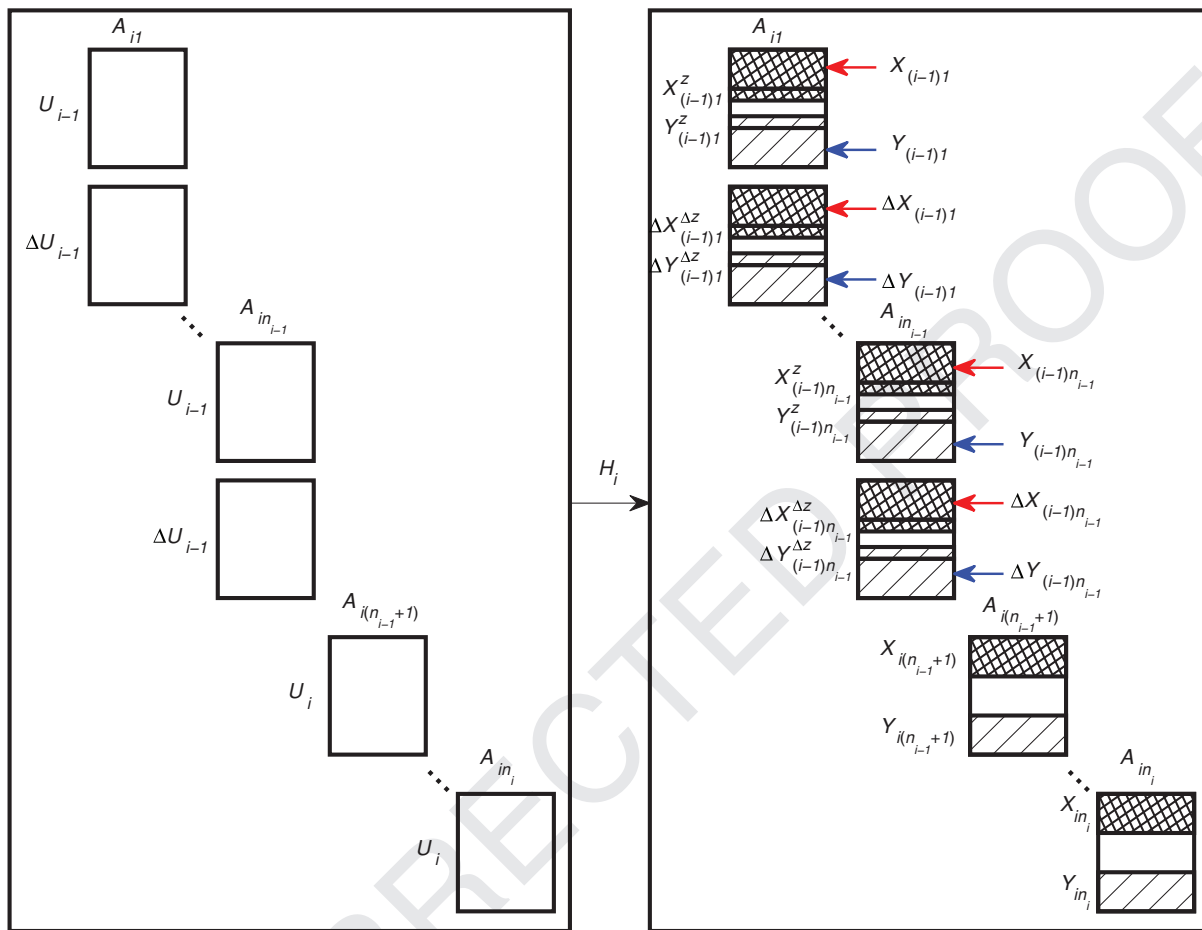


Fig. 4. The graph of  $\mathcal{H}_i$ .

317 Moreover, Algorithm 1 describes the detailed procedure of solving this problem. The time complexity is analyzed as  
 318 follows. Suppose that  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$  is the input three-way cognitive computing system. Running Step 1 takes  $O(|A_1|^2|U_1|)$   
 319 based on Eqs. (10) and (18). The time complexity of Steps 3–11 is  $O(|A_i|(|A_i|^2 + |U_i|))$ , and that of Steps 12–24 is  $O(|A_i|^2|U_i|)$ .  
 320 As a result, the time complexity of Algorithm 1 is  $O(n|A_n|^2(|A_n| + |U_n|))$ , where  $n$  is the number of three-way cognitive  
 321 computing states. Obviously, its time complexity is polynomial.

322 Finally, we use an example to illustrate Algorithm 1. In order to make the example better understood, we give below a  
 323 sufficient and necessary condition to the preparatory work of generating  $G_{\mathcal{H}_i \mathcal{L}_i}$ .

324 A three-way cognitive computing state  $\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i} = (G_{\mathcal{H}_{i-1} \mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$  can be constructed to obtain three-way granular  
 325 concepts  $G_{\mathcal{H}_i \mathcal{L}_i}$  via Eqs. (16)–(18) if and only if the following conditions are prepared:

- 326 (a) objects and attributes are updated;
- 327 (b) the evaluation function  $f_{A_i}$  and the thresholds  $\alpha$  and  $\beta$  are properly defined;
- 328 (c)  $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$  is satisfied;
- 329 (d) three-way granular concepts  $G_{\mathcal{H}_{i-1} \mathcal{L}_{i-1}}$  of the previous state are known;
- 330 (e) information granules of  $\mathcal{H}_{i-1}$ ,  $\mathcal{H}_{\Delta U_{i-1}}$  and  $\mathcal{H}_{\Delta A_{i-1}}$  are computed.

331 **Example 4.** In Example 1, nine manuscripts were evaluated by seven reviewers. As time goes by, on one hand, new  
 332 manuscripts will arrive. On the other hand, those falling into the boundary regions need to be further evaluated by inviting  
 333 additional reviewers. It is supposed that the information updating on the manuscripts and reviewers is shown in Table 2.  
 334 That is, new manuscripts  $x_{10}$ ,  $x_{11}$ ,  $x_{12}$ ,  $x_{13}$  and  $x_{14}$  were submitted to the reviewing dataset, and at the same time, new  
 335 reviewers  $r_8$ ,  $r_9$ ,  $r_{10}$  and  $r_{11}$  were invited to evaluate the manuscripts falling into the boundary regions. It can be observed  
 336 from Table 2 that the manuscript  $x_{10}$  falls into Domain 2, the manuscript  $x_{11}$  Domain 1, the manuscript  $x_{12}$  Domain 3, and  
 337 others a new domain (i.e. Domain 4). Among these newly invited reviewers,  $r_8$  is from Domain 1,  $r_9$  is from Domain 2,  
 338 and  $r_{10}$  and  $r_{11}$  are from Domain 4. Since new manuscripts and reviewers have been added, it is necessary to update the  
 339 three-way granular concepts which can be found in Example 3.

**Algorithm 1** Computing three-way granular concepts of a three-way cognitive computing system.

**Require:**  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$ , where  $\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i} = (G_{\mathcal{H}_{i-1} \mathcal{L}_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$  is a three-way cognitive computing state with  $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$ .

**Ensure:** Three-way granular concepts  $G_{\mathcal{H}_n \mathcal{L}_n}$  of  $\mathcal{F}$ .

```

1: Initialize  $G_{\mathcal{H}_1 \mathcal{L}_1} = \{(\mathcal{H}_1(A_{1s}), \mathcal{L}_1 \mathcal{H}_1(A_{1s})) \mid A_{1s} \in \mathcal{Q}(A_1)\}$  and  $i = 2$ ;
2: While  $i \leq n$ 
3:   Set  $\Omega_1 = \emptyset$ ;
4:   For each  $A_{it} \in \mathcal{Q}(A_i)$ 
5:     If there exists  $A_{i-1s} \in \mathcal{Q}(A_{i-1})$  such that  $A_{i-1s} \subseteq A_{it}$ 
6:       let  $\mathcal{H}_i(A_{it}) = \mathcal{H}_{i-1}(A_{i-1s}) \cup \mathcal{H}_{\Delta U_{i-1}}(A_{i-1s}) \cup \mathcal{H}_{\Delta A_{i-1}}(A_{it})$ ;
7:     Else
8:       let  $\mathcal{H}_i(A_{it}) = \mathcal{H}_{\Delta A_{i-1}}(A_{it})$ ;
9:     End If
10:    Set  $\Omega_1 \leftarrow \Omega_1 \cup \{\mathcal{H}_i(A_{it})\}$ ;
11:  End For
12:  Set  $\Omega_2 = \emptyset$ ;
13:  For each  $A_{it} \in \mathcal{Q}(A_i)$ 
14:    Let  $B = \emptyset$ 
15:    For each  $\mathcal{H}_i(A_{is}) \in \Omega_1$ 
16:      If  $\mathcal{H}_i(A_{is}) \preceq \mathcal{H}_i(A_{it})$ 
17:        do  $B \leftarrow B \cup \{A_{is}\}$ ;
18:      End If
19:    End For
20:    Set  $\mathcal{L}_i \mathcal{H}_i(A_{it}) = B$ ;
21:    Do  $\Omega_2 \leftarrow \Omega_2 \cup \{\mathcal{L}_i \mathcal{H}_i(A_{it})\}$ ;
22:  End For
23:  Compute  $G_{\mathcal{H}_i \mathcal{L}_i}$  based on  $\Omega_1$  and  $\Omega_2$ ;
24:   $i \leftarrow i + 1$ ;
25: End While
26: Return  $G_{\mathcal{L}_n \mathcal{H}_n}$ .
    
```

**Table 2**  
A reviewing dataset with information updating on manuscripts and reviewers.

$U_2$	Domain 1			Domain 2			Domain 3			Domain 4	
	$r_1$	$r_2$	$r_8$	$r_3$	$r_4$	$r_9$	$r_5$	$r_6$	$r_7$	$r_{10}$	$r_{11}$
$x_1$	Accept	Accept									
$x_2$	Accept	Reject	Reject								
$x_3$	Reject	Accept	Reject								
$x_4$				Accept	Reject	Accept					
$x_5$				Accept	Accept						
$x_6$				Reject	Reject						
$x_7$							Accept	Reject	Accept		
$x_8$							Accept	Accept	Accept		
$x_9$							Reject	Reject	Accept		
$x_{10}$				Accept	Accept						
$x_{11}$	Reject	Reject									
$x_{12}$							Reject	Accept	Reject		
$x_{13}$										Reject	Accept
$x_{14}$										Accept	Accept

Similar to the case in Example 1, we also take  $\alpha = \frac{2}{3}$  and  $\beta = \frac{1}{3}$ . The evaluation function  $f_{A_{it}}(x)$  ( $A_{it} \in \mathcal{Q}(A_i)$ ) is the ratio of the number of Accepts given to  $x$  to that of reviewers assigned to  $x$  under the columns  $A_{it}$ .

From Example 1, it follows  $U_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ ,  $A_1 = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$ ,  $A_{11} = \{r_1, r_2\}$ ,  $A_{12} = \{r_3, r_4\}$ ,  $A_{13} = \{r_5, r_6, r_7\}$  and  $\mathcal{Q}(A_1) = \{A_{11}, A_{12}, A_{13}\}$ . Let  $U_2 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}$ ,  $A_2 = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}\}$ ,  $A_{21} = \{r_1, r_2, r_8\}$ ,  $A_{22} = \{r_3, r_4, r_9\}$ ,  $A_{23} = \{r_5, r_6, r_7\}$ ,  $A_{24} = \{r_{10}, r_{11}\}$  and  $\mathcal{Q}(A_2) = \{A_{21}, A_{22}, A_{23}, A_{24}\}$ . Then,  $\mathcal{Q}(A_1) \leq \mathcal{Q}(A_2)$  is satisfied. Moreover, we denote  $\Delta U_1 = U_2 - U_1 = \{x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}$  and  $\Delta A_1 = A_2 - A_1 = \{r_8, r_9, r_{10}, r_{11}\}$ .

It can be seen from Example 3 that three-way granular concepts under  $\mathcal{H}_1$  and  $\mathcal{L}_1$  are as follows:

$$G_{\mathcal{H}_1 \mathcal{L}_1} = \{(\{x_1\}, \emptyset), \{A_{11}\}, (\{x_5\}, \{x_6\}), \{A_{12}\}, (\{x_7, x_8\}, \{x_9\}), \{A_{13}\}\}$$

348 from which we can obtain the information granules of  $\mathcal{H}_1$ .

349 Furthermore, we know from Table 2 that the information granules of  $\mathcal{H}_{\Delta U_1}$  are

$$\mathcal{H}_{\Delta U_1}^G = \{\{A_{11}\} \mapsto (\emptyset, \{x_{11}\}), \{A_{12}\} \mapsto (\{x_{10}\}, \emptyset), \{A_{13}\} \mapsto (\emptyset, \{x_{12}\})\},$$

350 and those of  $\mathcal{H}_{\Delta A_1}$  are

$$\mathcal{H}_{\Delta A_1}^G = \{\{A_{21}\} \mapsto (\emptyset, \{x_2, x_3\}), \{A_{22}\} \mapsto (\{x_4\}, \emptyset), \{A_{23}\} \mapsto (\emptyset, \emptyset), \{A_{24}\} \mapsto (\{x_{14}\}, \emptyset)\}.$$

351 To sum up, all the conditions (a)–(e) have been satisfied. According to the statement in the paragraph above Example 4, a  
352 three-way cognitive computing state  $\mathcal{F}_{\mathcal{H}_2\mathcal{L}_2} = (G_{\mathcal{H}_1\mathcal{L}_1}, \mathcal{H}_{\Delta U_1}, \mathcal{H}_{\Delta A_1})$  can be constructed to obtain three-way granular concepts  
353  $G_{\mathcal{H}_2\mathcal{L}_2}$  based on Eqs. (16)–(18). The detailed computations are given below:

$$\begin{aligned} \mathcal{H}_2(A_{21}) &= \mathcal{H}_1(A_{11}) \cup \mathcal{H}_{\Delta U_1}(A_{11}) \cup \mathcal{H}_{\Delta A_1}(A_{21}) = (\{x_1\}, \{x_2, x_3, x_{11}\}) \text{ due to } A_{11} \subseteq A_{21}, \\ \mathcal{H}_2(A_{22}) &= \mathcal{H}_1(A_{12}) \cup \mathcal{H}_{\Delta U_1}(A_{12}) \cup \mathcal{H}_{\Delta A_1}(A_{22}) = (\{x_4, x_5, x_{10}\}, \{x_6\}) \text{ due to } A_{12} \subseteq A_{22}, \\ \mathcal{H}_2(A_{23}) &= \mathcal{H}_1(A_{13}) \cup \mathcal{H}_{\Delta U_1}(A_{13}) \cup \mathcal{H}_{\Delta A_1}(A_{23}) = (\{x_7, x_8\}, \{x_9, x_{12}\}) \text{ due to } A_{13} \subseteq A_{23}, \\ \mathcal{H}_2(A_{24}) &= \mathcal{H}_{\Delta A_1}(A_{24}) = (\{x_{14}\}, \emptyset) \text{ since there does not exist } A_{1s} \subseteq A_{24} \text{ such that } A_{1s} \subseteq A_{24}. \end{aligned}$$

354 and

$$\mathcal{L}_2\mathcal{H}_2(A_{21}) = \{A_{21}\}, \mathcal{L}_2\mathcal{H}_2(A_{22}) = \{A_{22}\}, \mathcal{L}_2\mathcal{H}_2(A_{23}) = \{A_{23}\}, \mathcal{L}_2\mathcal{H}_2(A_{24}) = \{A_{24}\}.$$

355 Consequently, we have

$$G_{\mathcal{H}_2\mathcal{L}_2} = \{\langle (\{x_1\}, \{x_2, x_3, x_{11}\}), \{A_{21}\} \rangle, \langle (\{x_4, x_5, x_{10}\}, \{x_6\}), \{A_{22}\} \rangle, \langle (\{x_7, x_8\}, \{x_9, x_{12}\}), \{A_{23}\} \rangle, \langle (\{x_{14}\}, \emptyset), \{A_{24}\} \rangle\}.$$

356 Three-way decisions derived by the granular concepts  $G_{\mathcal{H}_2\mathcal{L}_2}$  are as follows:

- 357 •  $\langle (\{x_1\}, \{x_2, x_3, x_{11}\}), \{A_{21}\} \rangle$ : according to the reviewers  $r_1, r_2$  and  $r_8$  from Domain 1, manuscript  $x_1$  is accepted, while  $x_2,$   
358  $x_3$  and  $x_{11}$  are rejected;
- 359 •  $\langle (\{x_4, x_5, x_{10}\}, \{x_6\}), \{A_{22}\} \rangle$ : according to the reviewers  $r_3, r_4$  and  $r_9$  from Domain 2, manuscripts  $x_4, x_5$  and  $x_{10}$  are  
360 accepted, while  $x_6$  is rejected;
- 361 •  $\langle (\{x_7, x_8\}, \{x_9, x_{12}\}), \{A_{23}\} \rangle$ : according to the reviewers  $r_5, r_6$  and  $r_7$  from Domain 3, manuscripts  $x_7$  and  $x_8$  are accepted,  
362 while  $x_9$  and  $x_{12}$  are rejected;
- 363 •  $\langle (\{x_{14}\}, \emptyset), \{A_{24}\} \rangle$ : according to the reviewers  $r_{10}$  and  $r_{11}$  from Domain 4, manuscript  $x_{14}$  is accepted.

364 Furthermore, we point out that these three-way granular concepts can also be used in cognitive concept learning (see  
365 Examples 5 and 6 in the next section for details).

#### 366 4. Cognitive processes

367 From cognitive computing [42], the obtained three-way granular concepts of a three-way cognitive computing system  
368 can be further used to learn three-way cognitive concepts from a given clue. Note that the clue may be three-way decisions,  
369 a set of attribute classes or both of them. Generally speaking, deriving new cognitive concepts from a given clue by induc-  
370 tion, approximation or reasoning is called the cognitive process. For instance, we take Example 4 to describe the scenarios.  
371 Suppose that  $(\{x_1, x_4, x_5, x_7, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}\})$  is an available clue. Then what knowledge are such three-way deci-  
372 sions induced by? It is easy to observe from Example 4 that there is no a direct answer to this question since no extent of  
373 a three-way granular concept is exactly  $(\{x_1, x_4, x_5, x_7, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}\})$ . In what follows, we try to find the answers  
374 for this kind of questions.

375 Considering that the idea of lower and upper approximations in rough set theory has been widely applied to concept  
376 approximation [14,37,38], we use this idea to simulate cognitive processes.

377 In rough set theory [25], an information system is represented as  $I = (U, A)$  in which each object  $x \in U$  has a value  $a(x)$   
378 under every attribute  $a \in A$ .

379 For a nonempty subset  $A_i \subseteq A$ , an equivalence relation  $IND(A_i)$  is defined by

$$IND(A_i) = \{(x, y) \in U \times U \mid a(x) = a(y) \text{ for all } a \in A_i\}.$$

380 In fact,  $IND(A_i)$  can induce a partition  $U/IND(A_i)$  of  $U$  by taking each equivalence class as  $[x]_{A_i} = \{y \in U \mid (x, y) \in IND(A_i)\}$ .  
381 That is,  $U/IND(A_i) = \{[x]_{A_i} \mid x \in U\}$ . Then, for any target set  $X \subseteq U$ , its lower and upper approximations are respectively  
382 defined as

$$\underline{A}_i(X) = \bigcup_{Y \in U/IND(A_i), Y \subseteq X} Y \quad \text{and} \quad \overline{A}_i(X) = \bigcup_{Y \in U/IND(A_i), Y \cap X \neq \emptyset} Y,$$

383 and the pair  $[\underline{A}_i(X), \overline{A}_i(X)]$  is called a rough set of  $X$  with respect to  $A_i$ . The relationship between  $X$  and its lower and upper  
384 approximations is shown in Fig. 5.

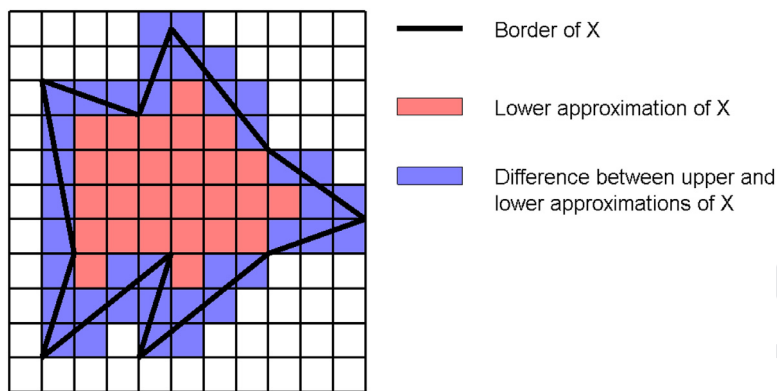


Fig. 5. The relationship among  $X$ ,  $\underline{A}_i(X)$  and  $\overline{A}_i(X)$ .

385 4.1. Three-way cognitive concept learning from three-way decisions

386 Based on the idea of lower and upper approximations, we put forward below an approach to learn an exact or two  
387 approximate three-way cognitive concepts from three-way decisions.

388 **Definition 7.** Let  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i, \mathcal{L}_i}\}$  be a three-way cognitive computing system and  $G_{\mathcal{H}_n, \mathcal{L}_n}$  be three-way granular concepts of  
389  $\mathcal{F}$ . Then, the lower and upper approximations of three-way decisions  $(X_0, Y_0)$  are respectively defined as

$$\underline{\text{Apr}}(X_0, Y_0) = \bigcup_{\langle (X,Y), B \rangle \in G_{\mathcal{H}_n, \mathcal{L}_n}, (X,Y) \preceq (X_0, Y_0)} (X, Y) \quad \text{and} \quad \overline{\text{Apr}}(X_0, Y_0) = \bigcup_{\langle (X,Y), B \rangle \in G_{\mathcal{H}_n, \mathcal{L}_n}, (X,Y) \cap (X_0, Y_0) \neq (\emptyset, \emptyset)} (X, Y). \quad (19)$$

390 **Proposition 6.** Both  $\underline{\text{Apr}}(X_0, Y_0)$  and  $\overline{\text{Apr}}(X_0, Y_0)$  are extents of three-way cognitive concepts.

391 **Proof.** Let  $\langle (X_i, Y_i), B_i \rangle, \langle (X_j, Y_j), B_j \rangle \in G_{\mathcal{H}_n, \mathcal{L}_n}$ . Then, by Eq. (8),  $\langle (X_i, Y_i) \cup (X_j, Y_j), \mathcal{L}_n \mathcal{H}_n(B_i \cup B_j) \rangle$  is a three-way cognitive con-  
392 cept, which means that  $(X_i, Y_i) \cup (X_j, Y_j)$  must be an extent of a three-way cognitive concept. Furthermore, by mathematical  
393 induction, we can prove that both  $\underline{\text{Apr}}(X_0, Y_0)$  and  $\overline{\text{Apr}}(X_0, Y_0)$  are extents of three-way cognitive concepts.  $\square$

394 **Definition 8.** For three-way decisions  $(X_0, Y_0)$ , we call

$$\langle \underline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\text{Apr}}(X_0, Y_0)) \rangle \quad \text{and} \quad \langle \overline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\text{Apr}}(X_0, Y_0)) \rangle$$

395 the learnt three-way cognitive concepts from  $(X_0, Y_0)$ . Moreover, the learning accuracy is defined as

$$\alpha(X_0, Y_0) = 1 - \frac{|\overline{\text{Apr}}(X_0, Y_0) - \underline{\text{Apr}}(X_0, Y_0)|}{2|U_n|},$$

396 where  $|\langle \cdot, \cdot \rangle|$  is the total number of elements in the first and the second sets.

397 From Definition 8, we know that  $\alpha(X_0, Y_0) = 1$  if and only if  $\overline{\text{Apr}}(X_0, Y_0) = \underline{\text{Apr}}(X_0, Y_0)$ . In this case, we can learn an exact  
398 three-way cognitive concept; otherwise, two approximate three-way cognitive concepts are learnt. Algorithm 2 gives the  
399 detailed procedure to learn cognitive concept(s) from three-way decisions.

400 According to Eq. (19), Steps 2–10 in Algorithm 2 are to compute the lower and upper approximations of three-way  
401 decisions  $(X_0, Y_0)$ . Furthermore, Step 11 is to find an exact or two approximate three-way cognitive concepts for  $(X_0, Y_0)$  as  
402 well as the learning accuracy  $\alpha(X_0, Y_0)$ . So, the time complexity of Algorithm 2 is  $O(|U_n||A_n|)$ .

403 **Example 5** (Continued with Example 4). Suppose that the manuscripts  $x_1, x_4, x_5, x_7$  and  $x_{10}$  were accepted, while  $x_2, x_3, x_6,$   
404  $x_9$  and  $x_{11}$  were rejected. Then which domain are the reviewers (making such three-way decisions) from? To answer this  
405 question, it needs to learn three-way cognitive concepts from  $X_0 = \{x_1, x_4, x_5, x_7, x_{10}\}$  and  $Y_0 = \{x_2, x_3, x_6, x_9, x_{11}\}$  based on  
406 the granular concepts  $G_{\mathcal{L}_2, \mathcal{H}_2}$ . By Eq. (19), it follows that:

$$\begin{aligned} \underline{\text{Apr}}(X_0, Y_0) &= \bigcup_{\langle (X,Y), B \rangle \in G_{\mathcal{H}_2, \mathcal{L}_2}, (X,Y) \preceq (X_0, Y_0)} (X, Y) \\ &= (\{x_1\}, \{x_2, x_3, x_{11}\}) \cup (\{x_4, x_5, x_{10}\}, \{x_6\}) \\ &= (\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}), \\ \overline{\text{Apr}}(X_0, Y_0) &= \bigcup_{\langle (X,Y), B \rangle \in G_{\mathcal{H}_2, \mathcal{L}_2}, (X,Y) \cap (X_0, Y_0) \neq (\emptyset, \emptyset)} (X, Y) \\ &= (\{x_1\}, \{x_2, x_3, x_{11}\}) \cup (\{x_4, x_5, x_{10}\}, \{x_6\}) \cup (\{x_7, x_8\}, \{x_9, x_{12}\}) \end{aligned}$$



**Algorithm 2** Three-way cognitive concept learning from three-way decisions.

**Require:** Three-way granular concepts  $G_{\mathcal{L}_n \mathcal{H}_n}$  of a three-way cognitive computing system  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$  and three-way decisions  $(X_0, Y_0)$ .

**Ensure:** An exact or two approximate three-way cognitive concepts with the learning accuracy for  $(X_0, Y_0)$ .

```

1: Initialize  $\Pi = \emptyset$ ,  $\Omega = \emptyset$ , and label the elements of  $G_{\mathcal{L}_n \mathcal{H}_n}$  as
    $\langle \mathcal{H}_n(A_{n1}), \mathcal{L}_n \mathcal{H}_n(A_{n1}) \rangle, \langle \mathcal{H}_n(A_{n2}), \mathcal{L}_n \mathcal{H}_n(A_{n2}) \rangle, \dots, \langle \mathcal{H}_n(A_{nt}), \mathcal{L}_n \mathcal{H}_n(A_{nt}) \rangle$ ;
2: For each  $i \in \{1, 2, \dots, t\}$ 
3:   If  $\mathcal{H}_n(A_{ni}) \preceq (X_0, Y_0)$ 
4:      $\Pi \leftarrow \Pi \cup \{\langle \mathcal{H}_n(A_{ni}), \mathcal{L}_n \mathcal{H}_n(A_{ni}) \rangle\}$ ;
5:   End If
6:   If  $\mathcal{H}_n(A_{ni}) \cap (X_0, Y_0) \neq (\emptyset, \emptyset)$ 
7:      $\Omega \leftarrow \Omega \cup \{\langle \mathcal{H}_n(A_{ni}), \mathcal{L}_n \mathcal{H}_n(A_{ni}) \rangle\}$ ;
8:   End If
9: End For
10: Set  $\underline{\text{Apr}}(X_0, Y_0) = \bigcup_{\langle (X, Y), B \rangle \in \Pi} (X, Y)$  and  $\overline{\text{Apr}}(X_0, Y_0) = \bigcup_{\langle (X, Y), B \rangle \in \Omega} (X, Y)$ ;
11: Compute  $\underline{B}_0 = \mathcal{L}_n(\underline{\text{Apr}}(X_0, Y_0))$ ,  $\overline{B}_0 = \mathcal{L}_n(\overline{\text{Apr}}(X_0, Y_0))$  and  $\alpha(X_0, Y_0) = 1 - \frac{|\overline{\text{Apr}}(X_0, Y_0) - \underline{\text{Apr}}(X_0, Y_0)|}{2|U_n|}$ ;
12: Return  $\langle \underline{\text{Apr}}(X_0, Y_0), \underline{B}_0 \rangle$ ,  $\langle \overline{\text{Apr}}(X_0, Y_0), \overline{B}_0 \rangle$  and  $\alpha(X_0, Y_0)$ .

```

$$= (\{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\}).$$

407 So, three-way cognitive concepts

$$\langle (\{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\}), \{A_{21}, A_{22}\} \rangle \text{ and } \langle (\{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\}), \{A_{21}, A_{22}, A_{23}\} \rangle$$

408 are learnt from  $(X_0, Y_0)$  with the learning accuracy  $\alpha(X_0, Y_0) = \frac{6}{7}$ . As a result, there does not exist any domain that the  
 409 reviewers making three-way decisions  $(X_0, Y_0)$  are from. However, the reviewers from Domains 1 and 2 made three-way  
 410 decisions which are decision-consistent with respect to  $(X_0, Y_0)$ , and  $(X_0, Y_0)$  is decision-consistent with respect to three-  
 411 way decisions made by the reviewers from Domains 1–3.

#### 412 4.2. Three-way cognitive concept learning from a set of attribute classes

413 Similar to the discussion in Section 4.1, we continue to learn an exact or two approximate three-way cognitive concepts  
 414 from a set of attribute classes.

415 **Definition 9.** Let  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$  be a three-way cognitive computing system and  $G_{\mathcal{H}_n \mathcal{L}_n}$  be three-way granular concepts of  
 416  $\mathcal{F}$ . Then, the lower and upper approximations of  $B_0 \in 2^{\mathcal{Q}(A_n)}$  are respectively defined as

$$\underline{\text{Apr}}(B_0) = \mathcal{L}_n \mathcal{H}_n \left( \bigcup_{\langle (X, Y), B \rangle \in G_{\mathcal{H}_n \mathcal{L}_n}, B \subseteq B_0} B \right) \text{ and } \overline{\text{Apr}}(B_0) = \mathcal{L}_n \mathcal{H}_n \left( \bigcup_{\langle (X, Y), B \rangle \in G_{\mathcal{H}_n \mathcal{L}_n}, B \cap B_0 \neq \emptyset} B \right). \quad (20)$$

417 **Proposition 7.** Both  $\underline{\text{Apr}}(B_0)$  and  $\overline{\text{Apr}}(B_0)$  are intents of three-way cognitive concepts.

418 **Proof.** The proof is obvious from Proposition 5 and Eq. (20).  $\square$

419 **Definition 10.** For any  $B_0 \in 2^{\mathcal{Q}(A_n)}$ , we call

$$\langle \mathcal{H}_n(\underline{\text{Apr}}(B_0)), \underline{\text{Apr}}(B_0) \rangle \text{ and } \langle \mathcal{H}_n(\overline{\text{Apr}}(B_0)), \overline{\text{Apr}}(B_0) \rangle$$

420 the learnt three-way cognitive concepts from  $B_0$ . Moreover, the learning accuracy is defined as

$$\beta(B_0) = 1 - \frac{|\overline{\text{Apr}}(B_0) - \underline{\text{Apr}}(B_0)|}{|A_n|}.$$

421 From Definition 10, we know that  $\beta(B_0) = 1$  if and only if  $\overline{\text{Apr}}(B_0) = \underline{\text{Apr}}(B_0)$ . In this case, we can learn an exact three-  
 422 way cognitive concept; otherwise, two approximate three-way cognitive concepts are learnt. Algorithm 3 gives the detailed  
 423 procedure to learn three-way cognitive concept(s) from a set of attribute classes.

424 According to Eq. (20), Steps 2–10 in Algorithm 3 are to compute the lower and upper approximations of  $B_0$ . Furthermore,  
 425 Step 11 is to find an exact or two approximate three-way cognitive concepts for  $B_0$  as well as the learning accuracy  $\beta(B_0)$ .  
 426 So, the time complexity of Algorithm 3 is  $O(|U_n||A_n|)$ .

**Algorithm 3** Three-way cognitive concept learning from a set of attribute classes.

**Require:** Three-way granular concepts  $G_{\mathcal{L}_n \mathcal{H}_n}$  of a three-way cognitive computing system  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$  and a set of attribute classes  $B_0$ .

**Ensure:** An exact or two approximate three-way cognitive concepts with the learning accuracy for  $B_0$ .

- 1: Initialize  $\Pi = \emptyset$ ,  $\Omega = \emptyset$ , and label the elements of  $G_{\mathcal{L}_n \mathcal{H}_n}$  as  $\langle \mathcal{H}_n(A_{n1}), \mathcal{L}_n \mathcal{H}_n(A_{n1}) \rangle, \langle \mathcal{H}_n(A_{n2}), \mathcal{L}_n \mathcal{H}_n(A_{n2}) \rangle, \dots, \langle \mathcal{H}_n(A_{nt}), \mathcal{L}_n \mathcal{H}_n(A_{nt}) \rangle$ ;
- 2: **For** each  $i \in \{1, 2, \dots, t\}$
- 3:   **If**  $\mathcal{L}_n \mathcal{H}_n(A_{ni}) \subseteq B_0$
- 4:      $\Pi \leftarrow \Pi \cup \{\langle \mathcal{H}_n(A_{ni}), \mathcal{L}_n \mathcal{H}_n(A_{ni}) \rangle\}$ ;
- 5:   **End If**
- 6:   **If**  $\mathcal{L}_n \mathcal{H}_n(A_{ni}) \cap B_0 \neq \emptyset$
- 7:      $\Omega \leftarrow \Omega \cup \{\langle \mathcal{H}_n(A_{ni}), \mathcal{L}_n \mathcal{H}_n(A_{ni}) \rangle\}$ ;
- 8:   **End If**
- 9: **End For**
- 10: Set  $\underline{\text{Apr}}(B_0) = \mathcal{L}_n \mathcal{H}_n \left( \bigcup_{\langle (X,Y), B \rangle \in \Pi} B \right)$  and  $\overline{\text{Apr}}(B_0) = \mathcal{L}_n \mathcal{H}_n \left( \bigcup_{\langle (X,Y), B \rangle \in \Omega} B \right)$ ;
- 11: Compute  $\langle X_0, Y_0 \rangle = \mathcal{H}_n(\underline{\text{Apr}}(B_0))$ ,  $\langle \bar{X}_0, \bar{Y}_0 \rangle = \mathcal{H}_n(\overline{\text{Apr}}(B_0))$  and  $\beta(B_0) = 1 - \frac{|\overline{\text{Apr}}(B_0) - \underline{\text{Apr}}(B_0)|}{|A_n|}$ ;
- 12: **Return**  $\langle \langle X_0, Y_0 \rangle, \underline{\text{Apr}}(B_0) \rangle, \langle \langle \bar{X}_0, \bar{Y}_0 \rangle, \overline{\text{Apr}}(B_0) \rangle$  and  $\beta(B_0)$ .

**Example 6** (Continued with **Example 4**). Which manuscripts were accepted and which ones were rejected by the reviewers  $r_1, r_2, r_3, r_4, r_8$  and  $r_9$  from Domains 1 and 2? Since  $A_{21} = \{r_1, r_2, r_8\}$  and  $A_{22} = \{r_3, r_4, r_9\}$ , then  $B_0 = \{A_{21}, A_{22}\}$  represents the reviewers under consideration. Moreover, to answer the above question, it needs to learn three-way cognitive concept(s) from  $B_0$  because there is no granular concept in  $G_{\mathcal{L}_2 \mathcal{H}_2}$  with its intent being  $B_0$  exactly. By **Eq. (20)**, we have

$$\begin{aligned} \underline{\text{Apr}}(B_0) &= \mathcal{L}_2 \mathcal{H}_2 \left( \bigcup_{\langle (X,Y), B \rangle \in G_{\mathcal{H}_2 \mathcal{L}_2}, B \subseteq B_0} B \right) \\ &= \mathcal{L}_2 \mathcal{H}_2(\{A_{21}\} \cup \{A_{22}\}) \\ &= \{A_{21}, A_{22}\}, \\ \overline{\text{Apr}}(B_0) &= \mathcal{L}_2 \mathcal{H}_2 \left( \bigcup_{\langle (X,Y), B \rangle \in G_{\mathcal{H}_2 \mathcal{L}_2}, B \cap B_0 \neq \emptyset} B \right) \\ &= \mathcal{L}_2 \mathcal{H}_2(\{A_{21}\} \cup \{A_{22}\}) \\ &= \{A_{21}, A_{22}\}. \end{aligned}$$

Thus, we find an exact three-way cognitive concept  $\langle \langle \{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\} \rangle, \{A_{21}, A_{22}\} \rangle$  from  $B_0$  with the learning accuracy  $\beta(B_0) = 1$ . In other words, the manuscripts  $x_1, x_4, x_5, x_{10}$  were accepted, while  $x_2, x_3, x_6$  and  $x_{11}$  were rejected by the reviewers from Domains 1 and 2.

#### 4.3. Three-way cognitive concept learning from three-way decisions and a set of attribute classes

In the previous **Sections 4.1** and **4.2**, we have discussed the case of learning three-way cognitive concepts from three-way decisions as well as a set of attribute classes. In the real world, however, it may be encountered that three-way decisions and a set of attribute classes are available simultaneously. This issue is investigated below.

**Definition 11.** Let  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$  be a three-way cognitive computing system and  $G_{\mathcal{H}_n \mathcal{L}_n}$  be three-way granular concepts of  $\mathcal{F}$ . For three-way decisions  $\langle X_0, Y_0 \rangle$  and  $B_0 \in 2^{\mathcal{Q}(A_n)}$ , if  $\underline{\text{Apr}}(X_0, Y_0) \preceq \mathcal{H}_n(B_0) \preceq \overline{\text{Apr}}(X_0, Y_0)$  and  $\underline{\text{Apr}}(B_0) \subseteq \mathcal{L}_n(X_0, Y_0) \subseteq \overline{\text{Apr}}(B_0)$ , we say that  $\langle X_0, Y_0 \rangle$  and  $B_0$  are jointly concept-inducible; otherwise, we say that they are jointly concept-uninducible.

**Definition 11** divides the pairs of  $\langle X_i, Y_i \rangle$  and  $B_i$  ( $i \in S$ ) into two categories: concept-inducible and concept-uninducible pairs. In what follows, we only discuss concept-inducible pairs since concept-uninducible ones are less related to each other.

**Example 7** (Continued with **Examples 5** and **6**). In **Example 5**,  $X_0 = \{x_1, x_4, x_5, x_7, x_{10}\}$ ,  $Y_0 = \{x_2, x_3, x_6, x_9, x_{11}\}$ ,  $\underline{\text{Apr}}(X_0, Y_0) = \langle \{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\} \rangle$  and  $\overline{\text{Apr}}(X_0, Y_0) = \langle \{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\} \rangle$ . By (iii) of **Definition 11**, we have  $\mathcal{L}_2(X_0, Y_0) = \{A_{21}, A_{22}\}$ . Moreover, in **Example 6**,  $B_0 = \{A_{21}, A_{22}\}$ ,  $\underline{\text{Apr}}(B_0) = \{A_{21}, A_{22}\}$ ,  $\overline{\text{Apr}}(B_0) = \{A_{21}, A_{22}\}$  and  $\mathcal{H}_2(B_0) = \langle \{x_1, x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\} \rangle$ . Then, according to **Definition 11**, we know that  $\langle X_0, Y_0 \rangle$  and  $B_0$  are jointly concept-inducible.

449 Now, we discuss how to learn three-way cognitive concepts from concept-inducible pairs.

450 **Definition 12.** Let  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$  be a three-way cognitive computing system and  $G_{\mathcal{H}_n \mathcal{L}_n}$  be three-way granular concepts  
451 of  $\mathcal{F}$ . For three-way decisions  $(X_0, Y_0)$  and  $B_0 \in 2^{\mathcal{Q}(A_n)}$ , if they are jointly concept-inducible, we call

$$\langle \underline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\text{Apr}}(X_0, Y_0)) \rangle \wedge \langle \mathcal{H}_n(\underline{\text{Apr}}(B_0)), \underline{\text{Apr}}(B_0) \rangle \quad (21)$$

452 and

$$\langle \overline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\text{Apr}}(X_0, Y_0)) \rangle \vee \langle \mathcal{H}_n(\overline{\text{Apr}}(B_0)), \overline{\text{Apr}}(B_0) \rangle \quad (22)$$

453 the learnt three-way cognitive concepts from the pair of  $(X_0, Y_0)$  and  $B_0$ . Furthermore, the learning accuracy is defined as

$$\gamma((X_0, Y_0), B_0) = \min\{\alpha(X_0, Y_0), \beta(B_0)\}.$$

454 From Definition 12, we know that  $\gamma((X_0, Y_0), B_0) = 1$  if and only if  $\overline{\text{Apr}}(X_0, Y_0) = \underline{\text{Apr}}(X_0, Y_0)$  and  $\overline{\text{Apr}}(B_0) = \underline{\text{Apr}}(B_0)$ . In  
455 this case, we learn an exact three-way cognitive concept; otherwise, two approximate three-way cognitive concepts are  
456 learnt. Algorithm 4 shows the detailed procedure to learn three-way cognitive concept(s) from three-way decisions and a  
set of attribute classes.

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**Algorithm 4** Cognitive concept learning from three-way decisions and a set of attribute classes.

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**Require:** Three-way granular concepts  $G_{\mathcal{L}_n \mathcal{H}_n}$  of a three-way cognitive computing system  $\mathcal{F} = \bigcup_{i=2}^n \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}\}$  and the pair of

$(X_0, Y_0)$  and  $B_0$ .

**Ensure:** Three-way cognitive concept(s) learnt from the concept-inducible pair of  $(X_0, Y_0)$  and  $B_0$ .

1: Call Algorithm 2 to learn  $\langle \underline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\text{Apr}}(X_0, Y_0)) \rangle, \langle \overline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\text{Apr}}(X_0, Y_0)) \rangle$  and  $\alpha(X_0, Y_0)$ , and Algorithm 3 to  
learn  $\langle \mathcal{H}_n(\underline{\text{Apr}}(B_0)), \underline{\text{Apr}}(B_0) \rangle, \langle \mathcal{H}_n(\overline{\text{Apr}}(B_0)), \overline{\text{Apr}}(B_0) \rangle$  and  $\beta(B_0)$ ;

2: **If**  $\underline{\text{Apr}}(X_0, Y_0) \preceq \mathcal{H}_n(B_0) \preceq \overline{\text{Apr}}(X_0, Y_0)$  or  $\overline{\text{Apr}}(B_0) \subseteq \mathcal{L}_n(X_0, Y_0) \subseteq \underline{\text{Apr}}(B_0)$  does not hold

3: **Return** “ $(X_0, Y_0)$  and  $B_0$  are jointly concept-uninducible”;

4: **Else**

5: do

$$\begin{aligned} \langle (X_1, Y_1), B_1 \rangle &\leftarrow \langle \underline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\underline{\text{Apr}}(X_0, Y_0)) \rangle \wedge \langle \mathcal{H}_n(\underline{\text{Apr}}(B_0)), \underline{\text{Apr}}(B_0) \rangle, \\ \langle (X_2, Y_2), B_2 \rangle &\leftarrow \langle \overline{\text{Apr}}(X_0, Y_0), \mathcal{L}_n(\overline{\text{Apr}}(X_0, Y_0)) \rangle \vee \langle \mathcal{H}_n(\overline{\text{Apr}}(B_0)), \overline{\text{Apr}}(B_0) \rangle, \\ \gamma((X_0, Y_0), B_0) &\leftarrow \min\{\alpha(X_0, Y_0), \beta(B_0)\}; \end{aligned}$$

6: **End If**

7: **Return**  $\langle (X_1, Y_1), B_1 \rangle, \langle (X_2, Y_2), B_2 \rangle$  and  $\gamma((X_0, Y_0), B_0)$ .

---

457

458 Based on the time complexity of Algorithms 2 and 3, we know that the time complexity of Algorithm 4 is  $O(|U_n||A_n|)$ .

459 **Example 8** (Continued with Example 7). It has been known from Example 7 that  $(X_0, Y_0) = (\{x_1, x_4, x_5, x_7, x_{10}\}, \{x_2, x_3, x_6, x_9,$   
460  $x_{11}\})$  and  $B_0 = \{A_{21}, A_{22}\}$  are jointly concept-inducible. Moreover, in Example 5, two approximate cognitive concepts  $\langle \{x_1,$   
461  $x_4, x_5, x_{10}\}, \{x_2, x_3, x_6, x_{11}\} \rangle, \langle \{A_{21}, A_{22}\} \rangle$  and  $\langle \{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\} \rangle, \langle \{A_{21}, A_{22}, A_{23}\} \rangle$  were learnt  
462 from  $(X_0, Y_0)$  with the learning accuracy  $\alpha(X_0, Y_0) = \frac{6}{7}$ . Additionally, in Example 6, an exact cognitive concept  $\langle \{x_1, x_4, x_5,$   
463  $x_{10}\}, \{x_2, x_3, x_6, x_{11}\} \rangle, \langle \{A_{21}, A_{22}\} \rangle$  was learnt from  $B_0$  with the learning accuracy  $\beta(B_0) = 1$ .

464 Then, based on Eqs. (21) and (22), we can learn two approximate three-way cognitive concepts  $\langle \{x_1, x_4, x_5, x_{10}\}, \{x_2,$   
465  $x_3, x_6, x_{11}\} \rangle, \langle \{A_{21}, A_{22}\} \rangle$  and  $\langle \{x_1, x_4, x_5, x_7, x_8, x_{10}\}, \{x_2, x_3, x_6, x_9, x_{11}, x_{12}\} \rangle, \langle \{A_{21}, A_{22}, A_{23}\} \rangle$  from  $(X_0, Y_0)$  and  $B_0$  with the  
466 learning accuracy  $\gamma((X_0, Y_0), B_0) = \frac{6}{7}$ . That is to say,  $(X_0, Y_0)$  and  $B_0$  are not completely matched with each other, but they  
467 can induce two approximate cognitive concepts with 86% accuracy. Moreover, the following decisions can be made by the  
468 induced approximate cognitive concepts:

- 469 • the reviewers from Domains 1–3 accepted the manuscripts  $x_1, x_4, x_5, x_7, x_8, x_{10}$ , but rejected  $x_2, x_3, x_6, x_9, x_{11}$  and  $x_{12}$ ;
- 470 • the reviewers from Domains 1 and 2 accepted the manuscripts  $x_1, x_4, x_5, x_{10}$ , but rejected  $x_2, x_3, x_6$  and  $x_{11}$ .

## 471 5. Numerical experiments

472 In this section, we conduct some numerical experiments to evaluate the performance of the proposed learning methods.

473 In the experiments, we chose five datasets from UCI Machine Learning Repository [8]: the Letter Recognition dataset,  
474 KEGG Metabolic Relation Network dataset, Skin Segmentation dataset, 3D Road Network dataset and Poker Hand dataset. The  
475 detailed information about these datasets is described in Table 3. In the experiments, the first attribute “Pathway text” in  
476 KEGG Metabolic Relation Network dataset was excluded since it is symbolic.

477 In order to generate standard datasets (i.e., their attributes are all Boolean), a data pre-processing technique was applied  
478 to the five chosen datasets. See Table 4 for the details, where “/” means “taking no action”, “Bisection” means “splitting the

**Table 3**

The detailed information about the five chosen datasets in the experiments.

Dataset	Instances	Attributes
Letter Recognition	20,000	16 (Discrete, each having 16 values)
KEGG Metabolic Relation Network	53,414	3 (Boolean), 20 (continuous)
Skin Segmentation	245,057	1 (Discrete, 2 values), 3 (continuous)
3D Road Network	434,874	4 (Continuous)
Poker Hand	1,025,010	11 (Discrete)

**Table 4**

Converting the five chosen datasets into standard datasets.

Dataset	Data pre-processing of attributes	Scaling
Letter Recognition	/	Nominal scale
KEGG Metabolic Relation Network	Bisection except the Boolean ones	Nominal scale
Skin Segmentation	Being divided into six equal segments except the discrete one	Nominal scale
3D Road Network	Being divided into six equal segments	Nominal scale
Poker Hand	Bisection	Nominal scale

**Table 5**

Designing three-way cognitive computing systems of the obtained standard datasets.

TWCCS	Design of parameters		
$\mathcal{F}^{(1)}$	$U_1 = \{1-2000\}, A_1 = \{1-25\},$ $U_4 = \{1-8000\}, A_4 = \{1-100\},$ $U_7 = \{1-14,000\}, A_7 = \{1-178\},$ $U_{10} = \{1-20,000\}, A_{10} = \{1-256\}$	$U_2 = \{1-4000\}, A_2 = \{1-50\},$ $U_5 = \{1-10,000\}, A_5 = \{1-125\},$ $U_8 = \{1-16,000\}, A_8 = \{1-204\},$	$U_3 = \{1-6000\}, A_3 = \{1-75\},$ $U_6 = \{1-12,000\}, A_6 = \{1-152\},$ $U_9 = \{1-18,000\}, A_9 = \{1-230\},$
$\mathcal{F}^{(2)}$	$U_1 = \{1-8902\}, A_1 = \{1-15\},$ $U_4 = \{1-35,608\}, A_4 = \{1-30\},$	$U_2 = \{1-17,804\}, A_2 = \{1-20\},$ $U_5 = \{1-44,510\}, A_5 = \{1-37\},$	$U_3 = \{1-26,706\}, A_3 = \{1-25\},$ $U_6 = \{1-53,414\}, A_6 = \{1-43\}$
$\mathcal{F}^{(3)}$	$U_1 = \{1-81,685\}, A_1 = \{1-10\},$	$U_2 = \{1-163,370\}, A_2 = \{1-15\},$	$U_3 = \{1-245,057\}, A_3 = \{1-20\}$
$\mathcal{F}^{(4)}$	$U_1 = \{1-144,958\}, A_1 = \{1-12\},$	$U_2 = \{1-289,916\}, A_2 = \{1-18\},$	$U_3 = \{1-434,874\}, A_3 = \{1-24\}$
$\mathcal{F}^{(5)}$	$U_1 = \{1-341,670\}, A_1 = \{1-10\},$	$U_2 = \{1-683,340\}, A_2 = \{1-15\},$	$U_3 = \{1-1,025,010\}, A_3 = \{1-22\}$

values of each attribute, from small to large, into two disjoint intervals whose lengths are the same”, and “Being divided into six equal segments” means “splitting the values of each attribute, from small to large, into six pairwise disjoint intervals whose lengths are the same”. Moreover, the scaling approach [46] was used to transform them into standard datasets. Here, we denote the obtained standard datasets by Datasets 1–5 which are in fact formal contexts.

Furthermore, Datasets 1, 2, 3, 4 and 5 were divided into segments for designing their corresponding three-way cognitive computing systems:  $\mathcal{F}^{(1)} = \bigcup_{i=2}^{10} \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}^{(1)}\}$ ,  $\mathcal{F}^{(2)} = \bigcup_{i=2}^6 \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}^{(2)}\}$ ,  $\mathcal{F}^{(3)} = \bigcup_{i=2}^3 \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}^{(3)}\}$ ,  $\mathcal{F}^{(4)} = \bigcup_{i=2}^3 \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}^{(4)}\}$  and  $\mathcal{F}^{(5)} = \bigcup_{i=2}^3 \{\mathcal{F}_{\mathcal{H}_i \mathcal{L}_i}^{(5)}\}$ , respectively. See Table 5 for the details, where TWCCS is the abbreviation of “Three-way cognitive computing system”. In the table,  $U_i = \{p-q\}$  means that  $U_i$  is constituted by the objects between the  $p$ th and  $q$ th objects including the endpoints, so does  $A_i$ . In addition, we show how  $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$  ( $i = 2, 3, \dots, 10$ ) were designed in  $\mathcal{F}^{(1)}$ . Specifically,  $\mathcal{Q}(A_{i-1}) = \{A_{(i-1)1}, A_{(i-1)2}, A_{(i-1)3}, A_{(i-1)4}, A_{(i-1)5}, A_{(i-1)6}\}$ , where the elements of each  $A_{(i-1)j}$  were taken from  $A_{i-1}$  in sequence. The cardinality of  $A_{(i-1)j}$  ( $j = 1, 2, 3, 4, 5$ ) is  $\lfloor \frac{|A_{i-1}|}{5} \rfloor$ , while that of  $A_{(i-1)6}$  is the remainder of  $|A_{i-1}|$  divided by 5. Similarly,  $\Delta A_{i-1} = A_i - A_{i-1} = \{\Delta A_{(i-1)1}, \Delta A_{(i-1)2}, \Delta A_{(i-1)3}, \Delta A_{(i-1)4}, \Delta A_{(i-1)5}, \Delta A_{(i-1)6}\}$ , where the elements of each  $\Delta A_{(i-1)j}$  were taken from  $\Delta A_{i-1}$  in sequence. The cardinality of  $\Delta A_{(i-1)j}$  ( $j = 1, 2, 3, 4, 5$ ) is  $\lfloor \frac{|\Delta A_{i-1}|}{5} \rfloor$ , while that of  $\Delta A_{(i-1)6}$  is the remainder of  $|\Delta A_{i-1}|$  divided by 5. Then  $\mathcal{Q}(A_i) = \{A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}, A_{i6}\}$  was defined by taking  $A_{ij} = A_{(i-1)j} \cup \Delta A_{(i-1)j}$ . As a result,  $\mathcal{Q}(A_{i-1}) \leq \mathcal{Q}(A_i)$  is satisfied. The cases of  $\mathcal{F}^{(2)}$ ,  $\mathcal{F}^{(3)}$ ,  $\mathcal{F}^{(4)}$  and  $\mathcal{F}^{(5)}$  were dealt with in a manner similar to  $\mathcal{F}^{(1)}$ , which is omitted here for convenience of presentation.

In the experiments, we took  $\alpha = \frac{3}{4}$  and  $\beta = \frac{1}{4}$ . Notice that the above standard datasets are formal contexts with the input data being ones and zeros. Then the evaluation function  $f_{B_i}(x)$  ( $B_i \in 2^{\mathcal{Q}(A_i)}$ ) was set to be the ratio of the number of ones given to  $x$  to that of ones and zeros given to  $x$  under the columns  $\cup B_i$ . Moreover, in order to guarantee the successful implementation of sequential three-way decisions, the information on the objects which had been classified into positive or negative regions in last cognitive computing state, was omitted when it comes into next cognitive computing state.

Then, Algorithm 1 was applied to Datasets 1–5. The corresponding running time is reported in Table 6, where  $|U|$  is the cardinality of object set,  $|A|$  is that of attribute set, and  $n$  is the number of three-way cognitive computing states. It can be seen from Table 6 that Algorithm 1 is reasonably efficient even for the largest dataset.

Using Algorithm 1, we have obtained the three-way granular concepts  $G_{\mathcal{L}_{10} \mathcal{H}_{10}}^{(1)}$ ,  $G_{\mathcal{L}_6 \mathcal{H}_6}^{(2)}$ ,  $G_{\mathcal{L}_3 \mathcal{H}_3}^{(3)}$ ,  $G_{\mathcal{L}_3 \mathcal{H}_3}^{(4)}$  and  $G_{\mathcal{L}_3 \mathcal{H}_3}^{(5)}$  of the three-way cognitive computing systems  $\mathcal{F}^{(1)}$ ,  $\mathcal{F}^{(2)}$ ,  $\mathcal{F}^{(3)}$ ,  $\mathcal{F}^{(4)}$  and  $\mathcal{F}^{(5)}$ . So, based on the theoretical results in Section 4, these granular concepts can be further used to learn three-way cognitive concepts from a given clue. Without loss of gener-

**Table 6**  
Experimental results.

Dataset	U	A	n	Running time(s)			
				Algorithm 1	Algorithm 2	Algorithm 3	Algorithm 4
Dataset 1	20,000	256	10	0.6843	0.0019	0.0021	0.0042
Dataset 2	53,414	43	6	1.1522	0.0022	0.0034	0.0064
Dataset 3	245,057	20	3	4.2865	0.0172	0.0123	0.2803
Dataset 4	434,874	24	3	12.8106	0.0180	0.0228	0.7907
Dataset 5	1,025,010	22	3	199.4306	0.0233	0.0297	0.8136

ality, we generated 100 clues randomly for Algorithm 2 as well as Algorithms 3 and 4. Then, Algorithms 2–4 were applied to Datasets 1–5. Here, Algorithms 2–4 were repeated 100 times since they are only able to achieve the learning task of a clue each time. The average running time of Algorithms 2–4 is also reported in Table 6. It can be observed from the table that they are all quite fast even for the largest dataset.

## 6. Final remarks

In this section, we give some remarks to conclude the paper.

(i) *A brief summary of our work.* To uncover the essential idea of three-way concepts for solving decision-making problems, we have discussed three-way cognitive concept learning via multi-granularity. Specifically, an axiomatic method of forming three-way cognitive concepts has firstly been proposed based on multi-granularity and three-way-decision-making principles. Then, a three-way cognitive computing system has been designed for learning composite three-way granular concepts. Moreover, cognitive processes have been simulated by the idea of low and upper approximations to learn three-way cognitive concepts from a given clue. Finally, numerical experiments have been conducted to evaluate the performance of the proposed learning methods.

(ii) *The significance of our research.* It is noticed that many different types of three-way concepts have been proposed in the existing literature and each of them has different properties. It is essential to identify which properties are intrinsic for characterizing three-way concepts in order to understand the basic decision-making mechanism of three-way concepts. Using multi-granularity and three-way-decision-making principles, our research has successfully clarified three properties which can be jointly used as axioms to characterize three-way concepts. In addition, as discussed in Section 2.2, these intrinsic properties have explicit semantics.

(iii) *The advantages of our methods.* We have designed a three-way cognitive computing system to learn granular concepts and proposed concept learning methods for simulating cognitive processes. Our three-way cognitive computing system can update three-way granular concepts as objects and attributes increase. What is more, the proposed concept learning methods can help to remember three-way cognitive concepts from a given clue. Besides, as shown by the experiments conducted in Section 5, our learning methods are quite efficient; they only take less than 200 seconds for the dataset with more than one million instances. Therefore, it seems possible for our methods to be applied in big data if some parallel computing techniques could be successfully developed.

(iv) *The differences and similarities between our study and the existing ones.* The idea of sequential three-way decisions has been adopted in this paper to establish an axiomatic method of forming three-way concepts. Granular computing has been incorporated into three-way cognitive concepts for constructing information granules, which guarantees that three-way granular concepts can be defined and used to remember new cognitive concepts from a given clue. What is more, the proposed three-way cognitive operators  $\mathcal{H}$  and  $\mathcal{L}$  form an isotone Galois connection between  $2^{Q(A)}$  and  $\mathcal{T}(U)$ . So, they are completely different from the classical cognitive operators [16] which form an antitone Galois connection between  $2^A$  and  $2^U$ . In other words, these two kinds of cognitive operators have different cognitive mechanisms.

Nevertheless, there are some similarities between our study and the existing ones. For instance, multi-granularity has been designed to be monotonous for supporting sequential three-way decisions, which was realized by  $Q(A_{i-1}) \leq Q(A_i)$  in the process of information updating. As usual, our sequential three-way decisions also become more and more effective from last three-way cognitive computing state to the next one. If information can be updated continually, the final result of our sequential three-way decisions will degenerate into two-way decisions (i.e., boundary regions disappear).

(v) *An outlook for further study.* Note that the classical cognitive operators have been reconsidered to fit the big data environment [14]. As a matter of fact, such a problem is also encountered in three-way cognitive operators. So, it is still necessary to redesign three-way cognitive operators for meeting different requirements of big data such as large-scale, multi-source and heterogeneous data. Moreover, in our opinion, cognitive logic should be introduced into three-way cognitive computing system for effectively simulating the human brain behaviors including learning, reasoning and so on. These issues will be investigated in our future work.

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